Second-Order Quantifier Elimination – Applications, Variations and Methods

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Second-Order Quantifier Elimination (SOQE)

Input: A formula **with second-order quantifiers**
Output: An **equivalent** formula **without second-order quantifiers**

$$\begin{aligned} \exists p\,((q \rightarrow p) \land (p \rightarrow r)) \quad &\equiv \quad ((q \rightarrow \bot) \land (\bot \rightarrow r)) \\ & \quad \lor\ ((q \rightarrow \top) \land (\top \rightarrow r)) \\ &\equiv \quad q \rightarrow r \end{aligned}$$

$$\exists p\,(\forall x\,(qx \rightarrow px) \land \forall x (px \rightarrow rx)) \quad \equiv \quad \forall x\,(qx \rightarrow rx)$$

$$\exists p\,((q \rightarrow pa) \land \forall x\,(px \rightarrow rx)) \quad \equiv \quad q \rightarrow ra$$

$$\exists p\,((q \rightarrow pa) \land (pb \rightarrow r)) \quad \equiv \quad a = b \rightarrow (q \rightarrow r)$$

SOQE Timeline

- 19th century: Algebra of Logic (Boole, Peirce, **Schröder**)
- 1910-20s: Early modern predicate logic (Löwenheim, Skolem, Behmann)
- **1935: Ackermann**
- 1950s: Craig
- **1990s: SCAN (Gabbay, Ohlbach), DLS (Doherty, Łukaszewicz, Szałas)**
- 2008: Monograph (Gabbay, Schmidt, Szałas)
- 2017: Workshop SOQE
- Major application areas in computational logic:
  - Modal correspondence
  - Circumscription
  - Description logic knowledge bases:
    Forgetting / uniform interpolation, modularization

Predicate Quantification: Basic Properties and Intuitions

- We split the set of all predicates into disjoint sets P and Q
- $\exists P\, F$ can be seen as **forgetting** about P in F
  - It expresses nothing about P

$$\mathrm{preds}(\exists P\, F) \subseteq Q$$

  - It expresses the same as F about Q

$$\text{If } \mathrm{preds}(G) \subseteq Q, \text{ then } F \models G \text{ iff } \exists P\, F \models G$$

Deciding Satisfiability

$$\exists P_1\, F \;\models\; \forall P_2\, G \quad \text{iff} \quad F \models G$$

if members of P_1, P_2 do not occur free

$$F \models \bot \quad \text{iff} \quad \exists P\, F \models \bot$$

- **Applying SOQE to** $\exists P F$ with $P = \mathrm{preds}(F)$ leads to $\top$, $\bot$, a formula with equality that constrains domain cardinality, or failure
- [*Behmann: Beitr. zur Alg. der Logik, insbes. zum Entscheidungsprob., 1922*]

Computing Correspondences

	$\forall p\,(\Box p \rightarrow p)$	**Hilbert-style axiom** T
$\equiv$	$\forall p \forall x\,(\forall y(rxy \rightarrow py) \rightarrow px)$	Standard translation
$\equiv$	$\neg\exists x \exists p(\forall y(rxy \rightarrow py) \wedge \neg px)$	
$\equiv$	$\neg\exists x\, \neg rxx$	SOQE
$\equiv$	$\forall x\, rxx$	**Frame condition** reflexivity

- Sahlqvist-van-Benthem algorithm
  see [*Blackburn et al.: Modal Logic, 2001*]
- SQEMA, ALBA, unified correspondence
  [*Conradie; Goranko; Vakarelov; Ghilardi; Palmigiano; Zhao 2006ff*]
- Proof theory, display logic
  [*Palmigiano; Conradie; Zhao 2017*]
- MQSEL, Ackermann approach
  [*Schmidt: The Ackermann approach for modal logic . . . , 2012*]

Circumscription

- Nonmonotonic reasoning by restricting semantics to **minimal models** [*McCarthy, Lifschitz, 1980s*]

The **circumscription** of predicate p in formula $F[p]$ can be defined as

$$F[p] \land \neg\exists p' (F[p'] \land p' < p)$$

where $p' < p$ stands for $\forall x\,(px \rightarrow p'x) \land \neg\forall x\,(p'x \rightarrow px)$

- **Computing circumscription** can be understood as SOQE [*Doherty, Łukaszewicz, Szałas: Computing circumscription revisited, 1997*]
- [Partial] stable models can be expressed in terms of circumscription [*Lin 1991; Janhunen et al. 1996; W 2010*] [*W: Abduction in logic programming as SOQE, 2013*]

Knowledge-Base Modularization

- [*Ghilardi, Lutz, Wolter: Did I damage my ontology?, 2006*]

G is a Q-**conservative extension** of F iff

$$G \models F \quad \text{and} \quad F \models \exists P\, G$$

- [*Grau et al.: Extracting Modules from Ontologies, 2009*]

F is a Q-**module** in G iff

$$G \models F \quad \text{and} \quad \text{for all } H \text{ with } \mathrm{preds}(H) \subseteq Q\text{:}\ \ G \models H \ \text{ iff } \ F \models H$$

F is a Q-module in G iff G is a Q-conservative extension of F

$\exists P\, G$ is the weakest Q-module in G

- [*Lutz, Wolter: Found. for unif. interpol. and forg. in expressive DLs, 2011*]
- [*Koopmann, Schmidt: Forg. Concept and Role Symb. in $\mathcal{ALCH}$-Ont., 2013*]

Craig Interpolation and Computation of Definientia

For first-order F, G such that $F \models G$ there exists a first-order H s.th.

1. $F \models H \models G$
2. $\mathrm{vocab}(H) \subseteq \mathrm{vocab}(F) \cap \mathrm{vocab}(G)$

- H can be extracted from a proof of $F \models G$

H is a **definiens** of G in F in terms of Q iff

1. $F \models \forall x\,(G \leftrightarrow H)$
2. $\mathrm{preds}(H) \subseteq Q$

H is a definiens of G in F in terms of Q iff

1. $\exists P\,(F \land G) \quad \models \quad H \quad \models \quad \neg\exists P\,(F \land \neg G)$
2. $\mathrm{preds}(H) \subseteq Q$

For first-order F, G definientia can constructed as Craig interpolants

Strongest Necessary and Weakest Sufficient Condition

H is a definiens of G in F in terms of Q iff

1. $\exists P\,(F \land G) \;\models\; H \;\models\; \neg\exists P\,(F \land \neg G)$
2. $\mathrm{preds}(H) \subseteq Q$

$\mathrm{SNC}_Q(F,G) \stackrel{\text{def}}{=} \exists P\,(F \land G)$

$\mathrm{WSC}_Q(F,G) \stackrel{\text{def}}{=} \neg\exists P\,(F \land \neg G)$

The strongest X in Q s.th. $F \land G \models X$

The weakest X in Q s.th. $F \land X \models G$

- Abduction, approximation

- [*Lin: On SN and WSCs, 2001*] (slightly different, not "semantical")
- [*Doherty, Łukaszewicz, Szałas: Comp. SN and WSCs of f.-o. formulas, 2001*]
- [*W: Projection and scope-determined circumscription, 2012*]

Consideration of Polarity, Forgetting about a Ground Atom

- For a **Craig-Lyndon interpolant** also the **polarity** of predicate occurrences is considered in vocab

  1. $F \models H \models G$
  2. $\mathrm{vocab}(H) \subseteq \mathrm{vocab}(F) \cap \mathrm{vocab}(G)$

- **Quantification upon a predicate in its negative polarity**

$$\exists -p\, F[p] \;\stackrel{\text{def}}{=}\; \exists p'\, (F[p'] \land \forall x\, (p'x \to px))$$

  - *[Lang, Liberatore, Marquis: Propositional Independence, 2003]*
  - *[W: Literal projection for first-order logic, 2008]*

- **Quantification upon a single ground atom**

$$\exists pa\, F[p] \;\stackrel{\text{def}}{=}\; \exists p'\, (F[p'] \land \forall x\, (x \neq a \to (p'x \leftrightarrow px))$$

  - *[Lin,Reiter: Forget it!, 1994]* Eliminable by expansion

$$\exists pa\, F[p] \;\equiv\; F[\lambda x.(x = a \lor px)] \;\lor\; F[\lambda x.(x \neq a \land px)]$$

Ackermann's Generalized Elimination Resultant

[*Ackermann: Untersuchungen über das Eliminationsproblem der mathematischen Logik, 1935*]

If F is universal, then $\exists p\, F$ is equivalent to a (possibly infinite) conjunction of a **recursive** set of first-order formulas

$$\begin{aligned}
& \neg\forall p\,[(pa \land \forall xy\,[(px \land sxy) \rightarrow py]) \rightarrow pb] \\
\equiv\;& \exists p\,(pa \;\land\; \neg pb \;\land\; \forall xy(\neg px \lor py \lor \neg sxy)) \\
\equiv\;& \quad a \neq b \\
& \land \neg sab \\
& \land \forall x_1\,(\neg sax_1 \lor \neg sx_1b) \\
& \land \forall x_1x_2\,(\neg sax_1 \lor \neg sx_1x_2 \lor \neg sx_2b) \\
& \land \forall x_1x_2x_3\,(\neg sax_1 \lor \neg sx_1x_2 \lor \neg sx_2x_3 \lor \neg sx_3b) \\
& \quad \vdots
\end{aligned}$$

Consequences in a Subvocabulary

[*Craig: Bases for first-order theories and subtheories, 1960*]

Find a **recursive** set of first-order formulas that has the **same first-order consequences** as $\exists P F$

- Weaker notion as equivalence [*Craig 1960*]
- Forgetting on the basis of an abstract "axiomatized" consequence operator [*Delgrande: A knowledge level account of forgetting, 2017*]

SOQE by Witnesses / Boolean Solution Problem / Boolean Unification

An *SOQE-witness* of P in $\exists P\, F[P]$

is a sequence $\boldsymbol{G}$ of formulas s.th. substitutible... (no free members of P) and

$$\exists P\, F[P] \equiv F[\boldsymbol{G}]$$

$F[\boldsymbol{G}]$ for $F[p_1 \mapsto \lambda x.G_1(x), \ldots, p_n \mapsto \lambda x.G_n(x)]$ followed by β-reducing

A *(particular) solution* of a *solution problem (SP)* $F[P]$

is a sequence $\boldsymbol{G}$ of formulas s.th. substitutible... (no free members of P) and

$$\models F[\boldsymbol{G}]$$

- *Auflösungsproblem* [*Schröder: Algebra der Logik, 1890–1905*]

- Finding a solution can also be understood as E-**unification** with constants in the theory of **Boolean algebra** (or logical equivalence)

- Cases related to **definability** that allow computation of solutions: p definable in F; p definable in $\neg F$; pa definable in $\neg F$; p nullary

The Boolean Solution Problem: Further Aspects

Given $F[p_1 \ldots p_n]$, find $G_1 \ldots G_n$ s.th. $\models F[G_1 \ldots G_n]$

- [*Schröder, Löwenheim, Behmann*]
- [*Rudeanu: Boolean Functions and Equations, 1974*]
- Boolean unification [*Baader 1998*]
- Quantifier-free [*Eberhard, Hetzl, Weller: Boolean unif. w. predicates, 2017*]
- [*W: The Boolean solution problem from the persp. of pred. logic, 2017*]

- Solutions of an **n-ary SP** can be expressed as solutions of **n unary SPs** on certain **existential second-order** formulas
  - Boole's method of successive eliminations

- Assume that a unary SP $F[p]$ has a solution iff $\models \exists p F[p]$:

For propositional logic: $\exists p\, F[p] \equiv F[F[\top]]$

$\exists P\, F[P]$ is the weakest formula X in which no member of P occurs free s.th.

$(X \rightarrow F[P])[P]$ has a solution

- **Reproductive solutions as most general solutions**

Basic SOQE Methods for Propositional Logic

- **"Expansion"**

$$\exists p\, F[p] \quad \equiv \quad \exists p\, ((p \land F[\top]) \lor (\neg p \land F[\bot])) \quad \equiv \quad F[\top] \lor F[\bot]$$

- **"Replace-by-resolvents"**

$$\begin{array}{ll} \exists p\ ((p \lor F) & \land \\ \quad (\neg p \lor G) & \land \\ \quad H) & \end{array} \quad \equiv \quad \begin{array}{l} (F \lor G)\ \land \\ H \end{array}$$

- **"Strike-out-from-DNF"**

$$\begin{array}{ll} \exists p\ ((p \land F) & \lor \\ \quad (\neg p \land G) & \lor \\ \quad H) & \end{array} \quad \equiv \quad \begin{array}{l} F\ \lor \\ G\ \lor \\ H \end{array}$$

Behmann's Method for Relational Monadic Formulas

- *[Behmann: Beitr. zur Alg. der Logik, insbes. zum Entscheidungsprob., 1922]*
- *[W: Heinrich Behmann's contributions to SOQE, 2015]*

Operates by **equivalence preserving formula rewriting**

1. Normalization of input $\exists p\, F$ to **"innex form"** w.r.t. first-order quantifiers
   - Propagate quantifiers inwards such that their **scopes do not overlap**
   - This is possible for RELMON and with **counting quantifiers** for RELMON with equality
   - Requires **expensive rewritings** such as distribution among $\wedge$ and $\vee$
   - Stronger than **antiprenexing**, e.g., *[Egly: 1994]*
2. Normalization such that **subformulas** $\exists p \ldots$ are in *Eliminationshauptform*
   - Involves potentially expensive inward propagation of $\exists p$
3. Conversion to match the left side of the **"Basic Elimination Lemma"**

$$\exists p\, (\forall x\, (F \vee px) \wedge \forall x\, (G \vee \neg px)) \;\; \equiv \;\; \forall x\, (F \vee G)$$

4. **Elimination by applying the Basic Elimination Lemma**
5. Postprocessing

Behmann's Method – A Look at some Internals

- Propagating quantifiers inwards such that their scopes do not overlap

$$
\begin{array}{ll}
 & \forall x\,(px \lor (qx \land \exists y\, ry)) \\
\equiv & \forall x\,((px \lor qx) \land (px \lor \exists y\, ry)) \\
\equiv & \forall x\,(px \lor qx) \land \forall z\,(pz \lor \exists y\, ry) \\
\equiv & \forall x\,(px \lor qx) \land (\forall z\, pz \lor \exists y\, ry)
\end{array}
$$

- The *Eliminationshauptform*, basis for $\exists p\,(\forall x\,(F \lor px) \land \forall x\,(G \lor \neg px))$

$$
\begin{array}{lll}
\exists p & (\bigwedge_{1 \leq i \leq a} \forall x\,(A_i[x] \lor px) & \land \\
 & \;\bigwedge_{1 \leq i \leq b} \forall x\,(B_i[x] \lor \neg px) & \land \\
 & \;\bigwedge_{1 \leq i \leq c} \exists x\,(C_i[x] \land px) & \land \\
 & \;\bigwedge_{1 \leq i \leq d} \exists x\,(D_i[x] \land \neg px)) &
\end{array}
$$

- Conversions might require introduction of equality

$$pt \quad \equiv \quad \forall x\,(x \neq t \lor px)$$

$$pt \quad \equiv \quad \exists x\,(x = t \land px)$$

Behmann: Decision Problem and Logic of Relations, 1926

- The case $\exists\varphi\forall x_1 \ldots x_n\, F$ can be reduced to the following sequence of problems:

  $\exists\varphi\forall x\, F[\varphi x, x]$,
  $\exists\varphi\chi\forall xy\, F[\varphi x, \chi y, x, y]$,
  $\exists\varphi\chi\psi\forall xyz\, F[\varphi x, \chi y, \psi z, x, y, z]$,
  $\ldots$

- Other prefixes can be handled by *Belegungsoperatoren* (idea ascribed to Schröder, Skolemization) that can only be undone in general by more entangled chainings of quantifiers (e.g., Henkin quantifiers)

Donnerstag, den 23. September, 9 Uhr vormittags.

I. Vorsitz: Schmitt.

H. Behmann, Halle: Entscheidungsproblem und Logik der Beziehungen.

Der Vortragende berichtet über einige Fortschritte in der Theorie des Entscheidungsproblems (vgl. diesen Jahresbericht, Bd. 32, S. *66—67*) für den Fall, daß neben veränderlichen Eigenschaften von Dingen auch veränderliche Beziehungen zwischen Dingen als Grundbestandteile zugelassen sind (während veränderliche Begriffe von Begriffen weiterhin ausgeschlossen bleiben), dessen Beherrschung bereits mancherlei mathematisch interessante Anwendungen, so

…

Indizes, die ihrerseits, wie der Vortragende zeigt, als „Belegungen" des Dingbereiches gedeutet werden können. Hiermit ergibt sich die Möglichkeit, die Elimination auch im allgemeinen Fall durchzuführen, wobei allerdings an Stelle des Begriffsoperators nunmehr Belegungsoperatoren auftreten, die sich im allgemeinen — wenigstens mit den bisherigen Mitteln — ihrerseits nicht rückgängig machen lassen. Dagegen gelingt ihre Beseitigung durch eine vom Vortragenden angegebene eigentümliche Erweiterung des Darstellungsschemas, die darauf hinausläuft, neben der bisher allein bekannten und verwendeten Aufeinanderfolge der Operationen „alle" und „es gibt" verwickeltere Ver-

Behmann – Notes on Entangled Chainings of Quantifiers, 1926

Ackermann → Behmann, 16 Aug 28

- *I believe to have thought through for some simple expressions of that kind that they can be replaced by no logically equivalent expression that is constructed only from individual quantifiers, α, β, γ and identity*
- *would be very grateful for clarification*

Behmann → Ackermann, 21 Aug 28

- *led the investigation to a certain point some years ago*
- *will send this to you as soon as I return*
- *became aware only between talk and correction*

Ackermann → Behmann, 1 Sep 28

- *took a load from my mind since I already struggled a lot with the elimination problem in the case where all individual quantifiers are universal*

Ackermann's Generalized Elimination Resultant

[Ackermann: Untersuchungen über das Eliminationsproblem der mathematischen Logik, 1935]

If F is universal, then $\exists p\, F$ is equivalent to a (possibly infinite) conjunction of a **recursive** set of first-order formulas

$$
\begin{aligned}
& \neg\forall p\,[(pa \land \forall xy\,[(px \land sxy) \rightarrow py]) \rightarrow pb] \\
\equiv\quad & \exists p\,(pa \;\land\; \neg pb \;\land\; \forall xy(\neg px \lor py \lor \neg sxy)) \\
\equiv\quad & \quad a \neq b \\
& \land \neg sab \\
& \land \forall x_1\,(\neg sax_1 \lor \neg sx_1b) \\
& \land \forall x_1x_2\,(\neg sax_1 \lor \neg sx_1x_2 \lor \neg sx_2b) \\
& \land \forall x_1x_2x_3\,(\neg sax_1 \lor \neg sx_1x_2 \lor \neg sx_2x_3 \lor \neg sx_3b) \\
& \quad \vdots
\end{aligned}
$$

SCAN ("Synthesizing Correspondence Axioms for Normal logics")

- *[Gabbay, Ohlbach: Quantifier elim. in second-order predicate logic, 1992]*

1. **Converting the input $\exists P\, F$ to CNF**, including Skolemization
2. **Applying C-resolution** (constraint-resolution) upon predicates in P
   - **Adding** non-redundant C-resolvents and C-factors

     $$\frac{C \lor ps \qquad \neg pt \lor D \quad \text{distinct clauses}}{C \lor D \lor s \neq t} \qquad \frac{C \lor ps \lor pt}{C \lor ps \lor s \neq t}$$

   - **Removing** clauses with a P-literal upon which no non-redundant inference can be performed
   - Might not terminate
3. **Unskolemization**
   - Might fail or lead just to Henkin quantifiers

- *[Engel: Quantifier elim. in second-order predicate logic, 1996]*
- `http://www.mettel-prover.org/scan/`
- *[Goranko et al.: SCAN is complete for all Sahlqvist formulae, 2004]*
- Variant for Herbrand semantics, atom patterns instead of constraints *[W: Semantic knowledge partitioning, 2004]*

Ackermann's Lemma

- Second-order quantification allows to understand the introduction and elimination of definitions as **equivalence preserving** transformations

$$\exists p\,(\forall x\,(px \leftrightarrow G) \land F[p]) \;\;\equiv\;\; F[G]$$

$F[G]$ for $F[p \mapsto \lambda x.G(x)]$ followed by β-reducing

- If p occurs in F only in a **single polarity**, then another result of [*Ackermann 1935*], **Ackermann's Lemma**, applies

$$\exists p\,(\forall x\,(px \rightarrow G) \land F[+p]) \;\;\equiv\;\; F[G]$$
$$\exists p\,(\forall x\,(px \leftarrow G) \land F[-p]) \;\;\equiv\;\; F[G]$$

DLS (Direct Methods, Ackermann Approach)

- [*Szałas: On the correspondence between modal and classical logic, 1993*]
- [*Doherty, Łukaszewicz, Szałas: Computing circumscription revisited, 1997*]

1. **Preprocessing** the input $\exists p\, F$ to obtain subformulas $\exists p\, (A[+p] \land B[-p])$
   - Might fail [W: If a formula is equivalent to $A[+p] \land B[-p]$, such conjuncts can be obtained by Craig-Lyndon interpolation]
2. **Preparation for Ackermann's Lemma**, for each such subformula
   - Might involve Skolemization
   - $\exists p\, (\forall x\, (px \rightarrow B') \land A''[+p])$ and $\exists p\, (\forall x\, (px \leftarrow A') \land B''[-p])$ should both be tried: need of Skolemization, size
3. **Application of Ackermann's Lemma**: Rewriting with $A''[B']$ or $B''[A']$, resp., and unskolemization
   - Unskolemization might fail or lead just to Henkin quantifiers
4. **Simplification**

- [*Gustafsson: An implementation ..., 1996*]
- [*Conradie: On the strength and scope of DLS, 2006*]

DLS – A Look at the Introduction of Skolemization

$$
\begin{aligned}
& \exists p\,(A[+p] \;\wedge\; B[-p]) \\
= \quad & \exists p\,(\forall x\,[(pax \vee pbx) \leftarrow qx] \;\wedge\; B) \\
\equiv \quad & \exists p\,(\forall x \exists u\,[\forall y\,(pyx \leftarrow [u = y \wedge qx]) \wedge (u = a \vee u = b)] \;\wedge\; B) \\
\equiv \quad & \exists f \exists p\,(\forall x\,[\forall y\,(pyx \leftarrow [fx = y \wedge qx]) \wedge (fx = a \vee fx = b)] \;\wedge\; B) \\
\equiv \quad & \exists f \exists p\,(\forall xy\,[pyx \leftarrow \underbrace{(fx = y \wedge qx)}_{A'}] \;\wedge\; \underbrace{\forall z\,[fz = a \vee fz = b] \wedge B}_{B''}) \\
= \quad & \exists f \exists p\,(\forall xy\,(pxy \leftarrow A') \;\wedge\; B''[-p])
\end{aligned}
$$

Fixpoint Generalization of Ackermann's Lemma, DLS*

- **Generalized Ackermann's Lemma for Fixpoint Logic**
  [*Nonnengart, Szałas: A fixpoint approach to SOQE, 1998*]

$$\exists p\, (\forall x\, (px \rightarrow G[+p]) \land F[+p]) \quad \equiv \quad F[\textsc{Gfp}\, px.G[+p]]$$
$$\exists p\, (\forall x\, (px \leftarrow G[+p]) \land F[-p]) \quad \equiv \quad F[\textsc{Lfp}\, px.G[+p]]$$

- Example

$$\begin{aligned} & \neg\exists z \exists p\, (\forall x\, (px \leftarrow (x = 0 \lor \exists y\, (syx \land py))) \land \neg pz) \\ \equiv \quad & \neg\exists z\, \neg\textsc{Lfp}\, pz.(z = 0 \lor \exists y\, (syz \land py)) \end{aligned}$$

► [*Koopmann, Schmidt: Uniform interp. of $\mathcal{ALC}$-ontol. using fixpoints, 2013*]

Ackermann's Quantifier Switching

- **"Ackermann's Quantifier Switching"**
  [*Ackermann: Zum Eliminationsproblem der mathematischen Logik, 1935*]

$$\exists p \forall x\, F[pxt_1, \ldots, pxt_n] \;\equiv\; \forall x\, \exists p'\, F[p't_1, \ldots, p't_n]$$

- Avoiding Skolemization, allowing monadic techniques, Ackermann's Lemma

- Right-to-left to achieve prenex form w.r.t. second-order quantifiers

- Can be applied to decide $\mathcal{ALC}$
  [*W: SOQE on relational monadic formulas, 2015*]

$$\begin{aligned} & \exists c_1 \ldots c_n\, \exists d_1 \ldots d_k\, \exists r_1 \ldots r_m\, \forall x\, F \\ \equiv\;\; & \exists c_1 \ldots c_n\, \exists d_1 \ldots d_k\, \forall x\, \exists r'_1 \ldots r'_m\, F[r_i xt \mapsto r'_i t] \end{aligned}$$

Craig's Setting: Approximating Resultants

- Instance-based theorem proving, Herbrand's theorem

$$H_1 \;=\!\!| \; H_2 \;=\!\!| \; H_3 \;=\!\!| \ldots =\!\!| \; \forall \boldsymbol{x}\, H$$

  If $\forall \boldsymbol{x}\, H \models \bot$, then there exists a $k \geq 1$ s.th. $H_k \models \bot$

- **An analogous setting for elimination**
  Adapted from [*Craig: Bases for First-Order Theories and Subtheories, 1960*]

From $\exists P\, F$ construct a **recursive** set of first-order formulas

$$G_i = \mathit{Prefix}_i\, \text{ELIM}(\exists P\, \mathit{Matrix}_i)$$

s.th.

$$G_1 \;=\!\!| \; G_2 \;=\!\!| \; G_3 \;=\!\!| \ldots =\!\!| \; \exists P\, F$$

and for all first-order H s.th. $\exists P\, F \models H$ there exists a $k \geq 1$ s.th. $G_k \models H$

$$\begin{array}{rcl} \exists P\, F & \models & \exists P\, \mathit{Prefix}_i\, \mathit{Matrix}_i \\ & \models & \mathit{Prefix}_i \exists P\, \mathit{Matrix}_i \\ & \equiv & \mathit{Prefix}_i\, \text{ELIM}(\exists P\, \mathit{Matrix}_i) \quad \models \quad H \end{array}$$

- A construction for universal F that leads to additional properties of the G_i
  [*W: Approximating resultants of existential SOQE, 2017*]

Summary

- **Applications**
  - Correspondences, Circumscription, Conservative extensions
- **Variations and Related Operations**
  - Craig interpolation, Definientia, SNC and WSC
  - Consideration of polarity and single ground atoms
  - Ackermann's generalized resultant, Consequences in a subvocabulary
  - Witnesses, Boolean solution problem, Boolean unification
- **Methods**
  - Propositional methods
  - Behmann's method for relational monadic formulas
  - The road to Ackermann's results
  - SCAN
  - Ackermann's lemma, DLS
  - Ackermann's quantifier switching
  - Craig's setting: approximating elimination

1. Introduction
2. Some Applications of SOQE
3. Variations of SOQE and Related Operations
4. SOQE Methods
5. Conclusion

References

References I

[Ackermann, 1935a] Ackermann, W. (1935a).
Untersuchungen über das Eliminationsproblem der mathematischen Logik.
Math. Ann., 110:390–413.

[Ackermann, 1935b] Ackermann, W. (1935b).
Zum Eliminationsproblem der mathematischen Logik.
Math. Ann., 111:61–63.

[Baader, 1998] Baader, F. (1998).
On the complexity of Boolean unification.
Inf. Process. Lett., 67(4):215–220.

[Baumgartner and Thorstensen, 2010] Baumgartner, P. and Thorstensen, E. (2010).
Instance based methods – A brief overview.
KI, 24(1):35–42.

[Behmann, 1922] Behmann, H. (1922).
Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem.
Math. Ann., 86(3–4):163–229.

References II

[Behmann, 1927] Behmann, H. (1927).
Entscheidungsproblem und logik der beziehungen.
In *Jahresbericht der Deutschen Mathematiker-Vereiningung*, volume 36, 2. Abt. Heft 1/4, pages 17–18.

[Behmann, 1950] Behmann, H. (1950).
Das Auflösungsproblem in der Klassenlogik.
Archiv für Philosophie, 4(1):97–109.
(First of two parts, also published in *Archiv für mathematische Logik und Grundlagenforschung*, 1.1 (1950), pp. 17-29).

[Behmann, 1951] Behmann, H. (1951).
Das Auflösungsproblem in der Klassenlogik.
Archiv für Philosophie, 4(2):193–211.
(Second of two parts, also published in *Archiv für mathematische Logik und Grundlagenforschung*, 1.2 (1951), pp. 33-51).

[Blackburn et al., 2001] Blackburn, P., de Rijke, M., and Venema, Y. (2001).
Modal Logic.
Cambr. Univ. Press.

References III

[Büttner and Simonis, 1987] Büttner, W. and Simonis, H. (1987).
Embedding Boolean expressions into logic programming.
J. Symb. Comput., 4(2):191–205.

[Conradie, 2006] Conradie, W. (2006).
On the strength and scope of DLS.
JANCL, 16(3–4):279–296.

[Craig, 1960] Craig, W. (1960).
Bases for first-order theories and subtheories.
JSL, 25(2):97–142.

[Delgrande, 2017] Delgrande, J. P. (2017).
A knowledge level account of forgetting.
JAIR, 60:1165–1213.

[Doherty et al., 1997] Doherty, P., Łukaszewicz, W., and Szałas, A. (1997).
Computing circumscription revisited: A reduction algorithm.
JAR, 18(3):297–338.

References IV

[Doherty et al., 2001] Doherty, P., Łukaszewicz, W., and Szałas, A. (2001).
Computing strongest necessary and weakest sufficient conditions of first-order formulas.
In *IJCAI-01*, pages 145–151.

[Eberhard et al., 2017] Eberhard, S., Hetzl, S., and Weller, D. (2017).
Boolean unification with predicates.
J. Logic and Comput., 27(1):109–128.

[Egly, 1994] Egly, U. (1994).
On the value of antiprenexing.
In *LPAR '94*, volume 822 of *LNCS*, pages 69–83.

[Engel, 1996] Engel, T. (1996).
Quantifier elimination in second-order predicate logic.
Master's thesis, Max-Planck-Institut für Informatik, Saarbrücken.

References V

[Fermüller et al., 2001] Fermüller, C., Leitsch, A., Hustadt, U., and Tammet, T. (2001).
Resolution decision procedures.
In Robinson, A. and Voronkov, A., editors, *Handb. of Autom. Reasoning*, volume 2, pages 1793–1849. Elsevier.

[Gabbay and Ohlbach, 1992] Gabbay, D. and Ohlbach, H. J. (1992).
Quantifier elimination in second-order predicate logic.
In *KR'92*, pages 425–435. Morgan Kaufmann.

[Gabbay et al., 2008] Gabbay, D. M., Schmidt, R. A., and Szałas, A. (2008).
Second-Order Quantifier Elimination: Foundations, Computational Aspects and Applications.
College Pub.

[Ghilardi et al., 2006] Ghilardi, S., Lutz, C., and Wolter, F. (2006).
Did I damage my ontology? A case for conservative extensions in description logics.
In *KR'06*, pages 187–197. AAAI Press.

References VI

[Goranko et al., 2004] Goranko, V., Hustadt, U., Schmidt, R. A., and Vakarelov, D. (2004).

SCAN is complete for all Sahlqvist formulae.

In *RelMiCS 7*, volume 3051 of *LNCS*, pages 149–162.

[Grau et al., 2009] Grau, B. C., Horrocks, I., Kazakov, Y., and Sattler, U. (2009).

Extracting modules from ontologies: A logic-based approach.

In *Modular Ontologies: Concepts, Theories and Techniques for Knowledge Modularization*, volume 5445 of *LNCS*, pages 159–186.

[Gustafsson, 1996] Gustafsson, J. (1996).

An implementation and optimization of an algorithm for reducing formulae in second-order logic.

Technical Report LiTH-MAT-R-96-04, Dept. of Mathematics, University of Linköping.

[Hintikka, 1998] Hintikka, J. (1998).

The Principles of Mathematics Revisited.

Cambr. Univ. Press.

References VII

[Hintikka, 2007] Hintikka, J. (2007).
Socratic Epistemology: Explorations of Knowledge-Seeking by Questioning.
Cambr. Univ. Press.

[Janhunen et al., 2006] Janhunen, T., Niemelä, I., Seipel, D., Simons, P., and You, J.-H. (2006).
Unfolding partiality and disjunctions in stable model semantics.
ACM Transactions on Computational Logic, 7(1):1–37.

[Koopmann et al., 2017] Koopmann, P., Rudolph, S., Schmidt, R. A., and Wernhard, C., editors (2017).
Proceedings of the Workshop on Second-Order Quantifier Elimination and Related Topics, SOQE 2017, volume 2013 of *CEUR Workshop Proceedings*. CEUR-WS.org.

[Koopmann and Schmidt, 2013a] Koopmann, P. and Schmidt, R. A. (2013a).
Forgetting concept and role symbols in $\mathcal{ALCH}$-ontologies.
In *LPAR-19*, volume 8312 of *LNCS (LNAI)*, pages 552–567.

[Koopmann and Schmidt, 2013b] Koopmann, P. and Schmidt, R. A. (2013b).
Uniform interpolation of $\mathcal{ALC}$-ontologies using fixpoints.
In *FroCoS 2013*, volume 8152 of *LNCS (LNAI)*, pages 87–102.

References VIII

[Lang et al., 2003] Lang, J., Liberatore, P., and Marquis, P. (2003).
Propositional independence – formula-variable independence and forgetting.
JAIR, 18:391–443.

[Lifschitz, 1994] Lifschitz, V. (1994).
Circumscription.
In Gabbay, D. M., Hogger, C. J., and Robinson, J. A., editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 3, pages 298–352. Oxford University Press.

[Lin, 1991] Lin, F. (1991).
A Study of Nonmonotonic Reasoning.
PhD thesis, Stanford University.

[Lin, 2001] Lin, F. (2001).
On strongest necessary and weakest sufficient conditions.
AI, 128:143–159.

[Lin and Reiter, 1994] Lin, F. and Reiter, R. (1994).
Forget It!
In *Working Notes, AAAI Fall Symp. on Relevance*, pages 154–159.

References IX

[Lutz and Wolter, 2011] Lutz, C. and Wolter, F. (2011).
Foundations for uniform interpolation and forgetting in expressive description logics.
In *IJCAI-11*, pages 989–995. AAAI Press.

[Löwenheim, 1908] Löwenheim, L. (1908).
Über das Auflösungsproblem im logischen Klassenkalkül.
In *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, volume 7, pages 89–94. B. G. Teubner.

[Löwenheim, 1910] Löwenheim, L. (1910).
Über die Auflösung von Gleichungen im logischen Gebietekalkül.
Math. Ann., 68:169–207.

[Martin and Nipkow, 1986] Martin, U. and Nipkow, T. (1986).
Unification in Boolean rings.
In *CADE-8*, volume 230 of *LNCS (LNAI)*, pages 506–513.

[McCarthy, 1980] McCarthy, J. (1980).
Circumscription – a form of non-monotonic reasoning.
AI, 13:27–39.

References X

[Nonnengart and Szałas, 1998] Nonnengart, A. and Szałas, A. (1998).
A fixpoint approach to second-order quantifier elimination with applications to correspondence theory.
In Orlowska, E., editor, *Logic at Work. Essays Ded. to the Mem. of Helena Rasiowa*, pages 89–108. Springer.

[Rudeanu, 1974] Rudeanu, S. (1974).
Boolean Functions and Equations.
Elsevier.

[Rudeanu, 2001] Rudeanu, S. (2001).
Lattice Functions and Equations.
Springer.

[Schmidt, 2012] Schmidt, R. A. (2012).
The Ackermann approach for modal logic, correspondence theory and second-order reduction.
JAL, 10(1):52–74.

References XI

[Szalas, 1993] Szalas, A. (1993).
On the correspondence between modal and classical logic: An automated approach.
JLC, 3(6):605–620.

[Wernhard, 2004] Wernhard, C. (2004).
Semantic knowledge partitioning.
In *JELIA 04*, volume 3229 of *LNCS (LNAI)*, pages 552–564.

[Wernhard, 2008] Wernhard, C. (2008).
Literal projection for first-order logic.
In *JELIA 08*, volume 5293 of *LNCS (LNAI)*, pages 389–402.

[Wernhard, 2010] Wernhard, C. (2010).
Circumscription and projection as primitives of logic programming.
In *Tech. Comm. ICLP'10*, volume 7 of *LIPIcs*, pages 202–211. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

References XII

[Wernhard, 2012] Wernhard, C. (2012).
Projection and scope-determined circumscription.
J. Symb. Comput., 47(9):1089–1108.

[Wernhard, 2013] Wernhard, C. (2013).
Abduction in logic programming as second-order quantifier elimination.
In *FroCoS 2013*, volume 8152 of *LNCS (LNAI)*, pages 103–119.

[Wernhard, 2014] Wernhard, C. (2014).
Second-order characterizations of definientia in formula classes.
Technical Report KRR 14–03, TU Dresden.

[Wernhard, 2015a] Wernhard, C. (2015a).
Heinrich Behmann's contributions to second-order quantifier elimination.
Technical Report KRR 15–05, TU Dresden.

[Wernhard, 2015b] Wernhard, C. (2015b).
Second-order quantifier elimination on relational monadic formulas – A basic method and some less expected applications.
In *TABLEAUX 2015*, volume 9323 of *LNCS (LNAI)*.

References XIII

[Wernhard, 2017a] Wernhard, C. (2017a).

Approximating resultants of existential second-order quantifier elimination upon universal relational first-order formulas.

In *Proceedings of the Workshop on Second-Order Quantifier Elimination and Related Topics, SOQE 2017*, volume 2013 of *CEUR Workshop Proceedings*, pages 82–98. CEUR-WS.org.

[Wernhard, 2017b] Wernhard, C. (2017b).

The Boolean solution problem from the perspective of predicate logic.

In *FroCoS 2017*, volume 10483 of *LNCS (LNAI)*, pages 333–350.

[Wernhard, 2017c] Wernhard, C. (2017c).

The Boolean solution problem from the perspective of predicate logic – extended version.

Technical Report KRR 17-01, TU Dresden.