# Inspecting Gödel's Ontological Proof

## "Literate automated theorem proving" document created with *PIE*

## – Draft –

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# Contents

## Remark: Current Status of this Document

This is a draft of inspecting Gödel's ontological proof with the *PIE* (*Proving, Interpolating, Eliminating*) system. It gives an example of applying the system's "literate automated theorem proving" interface to formalize and investigate a nontrivial theory. The source code demonstrates for several recently added system features how these can be used. With respect to the subject, the analysis of Gödel's proof, this version of the document is to be seen just a very first draft. Nevertheless, some aspects might already be of interest, in particular those made possible through second-order quantifier elimination such as "reduced" views on *essence* and *necessary existence* as well as approaches to find weakest sufficient frame conditions.

# 1 Introduction

## 1.1 Background

Gödel bequeathed a short text with an ontological proof of the existence of God. In 1970 he showed the proof to Scott, who also recorded it in a slightly different version. Transcripts of both handwritten manuscripts have been published later by Sobel [Sob87]. From this starting point, a number of variations of Gödel's axiomatization have since been suggested in the literature. Comprehensive background and discussion is provided in Sobel's book [Sob04], which also reproduces the transcripts, and in Fitting's book [Fit02]. Both books present formalizations in modal predicate logic, along with formal proofs, in a natural deduction system and in the framework of analytic tableaux, respectively.

The investigation of Gödel's proof with automated systems was initiated by Benzmüller and Woltzenlogel Paleo in [BW14]. A higher-order modal logic is embedded there into classical higher-order logic, which, in turn, is supported by a combination of automated theorem proving and verification systems. In several follow-up works variations of Gödel's proof have been analyzed with different techniques and automated systems (see [KBZ19] for an overview). The automated approach enforces precise and detailed formalizations. Together with the possibility to test for vast numbers of combinations of axioms whether they entail candidate theorems this led to many new observations.

Here we approach Gödel's proof with an automated system that is centered round *first-order* theorem proving, which it extends by second-order quantifier elimination and the support for expressing first-order formalizations by means of schemas or macros. The impact of these techniques on the analysis of axiomatizations and proofs is can be summarized as follows:

**Classical First-Order Logic as Basis.** Compared to a higher-order setting, immediate limitations are that quantification upon predicate symbols is not permitted, predicates are not allowed to occur in argument position, and there is no abstraction mechanism that allows to construct predicates from formulas.[1] The first aspect, quantification upon predicates, is supported to some degree in our framework with second-order quantifier elimination, discussed below. The other aspects, predicates as arguments and construction of predicates through abstraction are in Gödel's proof actually only required with respect to specific instances that can be expressed in first-order logic. A potential reward for the explicit creation of instances is that information about which instances are used in proofs is then trivially available. Explicit instantiation by predicates and in some contexts also individuals suggests to use first-order logic together with schemas, as common in mathematics. Our framework supports this approach with a mechanism to specify formula macros. We represent modal formulas directly in their standard translation, which facilitates the consideration of frame conditions that are represented directly by first- or second-order formulas. First-order logic is well-known, ensuring that the results of investigations do not reflect unnoticed features of some special underlying logic.

**Second-Order Quantifier Elimination.** Second-order quantifier elimination [GSS08] is the computational task of computing for a given second-order formula an equivalent first-order formula. Since not all second-order formulas have a first-order equivalent, this task is inherently incomplete. A traditional application field of second-order quantifier elimination is to compute from a given modal axiom the corresponding frame property. Consider, for example, the axiom $\Box p \rightarrow p$, known as $M$ or $T$. Its correspondence to reflexivity of the accessibility relation $\mathsf{r}$ can be automatically established by second-order quantifier elimination:

Input: $\forall p\, \forall v\, (\forall w\, (\mathsf{r}(v,w) \rightarrow p(w)) \rightarrow p(v))$.
Result of elimination:

$$\forall x\, \mathsf{r}(x,x).$$

The elimination result extracts from the modal axiom what it states about the accessibility relation. In general, the extraction of knowledge about a subvocabulary by second-order quantifier elimination can be useful to gain insight into the meaning of axioms and defined concepts.

**Computing Weakest Sufficient Conditions.** The *weakest sufficient condition* [Lin01, DŁS01, Wer12] of formula $G$ on a set $Q$ of predicates within a formula $F$ can be char-

[1] Similar remarks also hold for functions in addition to predicates.

acterized as the second-order formula

$$\forall p_1 \ldots \forall p_n \, (F \rightarrow G),$$

where $p_1, \ldots, p_n$ are all predicates that occur free in $F \rightarrow G$ and are not members of $Q$. This second-order formula denotes the weakest (with respect to entailment) formula $H$ in which only predicates in $Q$ occur free such that $F \land H \rightarrow G$ is valid, or equivalently, such that $H \rightarrow (F \rightarrow G)$ is valid. Second-order quantifier elimination can be applied to "compute" a weakest sufficient condition, that is, converting it to a first-order formula, which, of course, is inherently not possible in all cases. This application pattern of second-order quantifier elimination seems particularly useful in the inspection of theories, as it allows to characterize in a backward, goal-oriented or abductive way the requirements about predicates $Q$ that are missing to conclude from some given axioms $F$ a given theorem $G$.

## 1.2 Technical Notes

1. This document is processed by the *PIE* system, described in [Wer16]. The formal macro definitions are read by the system. Macros without parameters play the role of formula names. The system invokes reasoners on proving, elimination and interpolation tasks. Their outputs are presented with phrases such as *This formula is valid*, *This formula is not valid*, and *Result of elimination.*

2. We write formulas of modal predicate logic as formulas of classical first-order logic by applying the standard translation from [vB10, Sec. 11.4] and [vB83, Chap. XII]. The binary predicates $\mathsf{r}$ and $\mathsf{e}$ are used for world accessibility and membership in the domain of a world.

3. As target logic we do not use a two-sorted logic nor encode two-sortedness explicitly with relativizer predicates. However, the translation of modal formula yields formulas in which all quantifications are relativized by $\mathsf{r}$ or by $\mathsf{e}$, which seems to subsume the effect of such relativizer predicates. To express that free individual symbols are of sort *world* we use the unary predicate $\mathsf{world}$. Macros 1 and 2 defined below can be used as axioms that relate $\mathsf{world}$ and $\mathsf{r}$ as far as needed here.

4. The used standard translation realizes with respect to the represented modal logic *varying domain semantics* (*actualist notion of quantification*), expressed with the existence predicate $\mathsf{e}$. Axioms that state domain increase and decrease can be used to to obtain *constant domain semantics* (*possibilist notion of quantification*).

5. As technical basis for Gödel's proof we use the presentation of Scott's version [Sob04, Chapter IV, Appendix B] in [BWW17, Fig. 1]. The axiom and theorem numbering follow these documents. In [BWW17, Fig. 1] there are two additional lemmas, *L1* and *L2*. Of these, we only use *L1* and call it *Lemma 2*, reserving *Lemma 1* for another lemma, used in an earlier proof stage.

6. The LaTeX presentation of formulas and macro definitions bears some footprint from Prolog's syntax, since the underlying system *PIE* uses a Prolog-based syntax for logical formulas and supports interaction through the Prolog interpreter: Macro parameters and bound logical variables that are to be bound to fresh symbols at macro expansion are represented in the system by Prolog variables, and thus start with capital letters. Where-clauses in macro definitions are used to display in abstracted form special Prolog code that is executed at macro expansion.

7. The available automated deduction techniques include the following:

   - First-order theorem proving, in particular with resolution/paramodulation (*Prover9*) and clausal tableaux (*CM*), as well as finding finite first-order "countermodels" (*Mace4*).[2] The clausal tableau prover is weak with equality, as it operates in a goal-oriented way, sometimes quite sensitive to settings like the particular division of a problem into axiom and theorem part, and has no means to ensure that a problem is (counter-) satisfiable like *Mace4*. It outputs clausal tableaux that can be graphically displayed.
   - Second-order quantifier elimination with an implementation of the *DLS* algorithm [DŁS97] that is based on Ackermann's Lemma.
   - Various methods for formula simplification, clausification and unskolemization that are applied in preprocessing, inprocessing, and for output presentation. (The latter seems a major issue by itself that is far from being solved.)
   - First-order Craig interpolation on the basis of clausal tableaux (currently not used in this document).

### 1.3 Structure of the Document

Sections 2–6 each discuss a stage of Gödel's argument, roughly following the division in [Fit02, Chapter 11]. Further aspects and variants are discussed in Sections 10–10. Section 11 is for auxiliary definitions of merely technical system related character. Observations that seem to be of particular interest for further investigation are highlighted with "►".

## 2 Positiveness

### 2.1 Auxiliary Sort Inference Predicate

To express that free individual symbols are of sort *world* we use the unary predicate $\mathsf{world}$. The following formulas can be used as axioms that leads from $\mathsf{r}(v, w)$ to $\mathsf{world}(v)$ and $\mathsf{world}(w)$ or, just to $\mathsf{world}(w)$, respectively. The latter, weaker, formula is sufficient in some of the considered contexts.

[2]Other first-order systems that support the TPTP format as well as propositional systems that support the DIMACS format could also be integrated.

1. *r_world*

Defined as

$$\forall v \forall w\, (\mathsf{r}(v, w) \rightarrow \mathsf{world}(v) \wedge \mathsf{world}(w)).$$

2. *r_world₁*

Defined as

$$\forall v \forall w\, (\mathsf{r}(v, w) \rightarrow \mathsf{world}(w)).$$

## 2.2 Representing Verum and Falsum

*Positiveness* is in Gödel's theory a predicate that applies to predicates. In the actual proof, however, it is used only in a small number of instances with specific argument predicates: $\lambda x.x = x$, $\lambda x.x \neq x$, and an arbitrary but fixed predicate. In correspondence with the standard translation, we represent $\lambda x.x = x$ and $\lambda x.x \neq x$ by binary predicates $\top$ and $\bot$, where the first argument is a world. These predicates may be defined follows:[3]

3. *topbot_def*

Defined as

$$\begin{array}{ll} \forall v \forall x\, (\mathsf{world}(v) \rightarrow (\top(v, x) \leftrightarrow \mathsf{e}(v, x))) & \wedge \\ \forall v \forall x\, (\mathsf{world}(v) \rightarrow (\bot(v, x) \leftrightarrow \neg \mathsf{e}(v, x))). & \end{array}$$

The following formula expresses equivalence of the binary predicate $\top$ and $\lambda vx.\neg\bot(v, x)$:

4. *topbot_equiv*

Defined as

$$\forall v \forall x\, (\mathsf{world}(v) \rightarrow (\top(v, x) \leftrightarrow \neg\bot(v, x))).$$

This formula is valid: *topbot_def* $\rightarrow$ *topbot_equiv*.

In our first-order framework we do not admit a predicate that has a predicate as argument. But for the purpose of Gödel's proof, this can be simulated it by a predicate

[3]Perhaps there are other possibilities to define them. The interplay of these predicates with the existence predicate seems not straightforward.

that is applied instead to an individual constant representing the argument predicate. We use the constants $\ulcorner\top\urcorner$, $\ulcorner\bot\urcorner$ and $\ulcorner\neg\top\urcorner$ to designate the individuals associated with $\top$, $\bot$ and $\lambda vx.\neg\top(v,x)$, respectively. The following axiom leads from the equivalence expressed by Macro 4 to equality of the associated individuals $\ulcorner\bot\urcorner$ and $\ulcorner\neg\top\urcorner$:

---

5. *topbot_equiv_equal*

---

Defined as

$$topbot\_equiv \rightarrow \ulcorner\bot\urcorner = \ulcorner\neg\top\urcorner.$$

Equality is here understood with respect to first-order logic, not qualified by a world parameter. In Section 9 below an alternative is shown, where in essence the equality is replaced by a weaker substitutivity property.

## 2.3 Proving Theorem 1

The left-to-right direction of Axiom 1 of Scott's version is rendered by the following macro. (The right-to-left direction is stated below as Macro 21.) We represent *is positive* by the binary predicate pos which has a world and an individual representing a predicate as argument.

At macro expansion, the individual constants $P'$ and $N'$ associated with the supplied predicate symbol $P$ and with $\lambda vx.\neg P(v,x)$, respectively, are determined by the code in the where clause. This technique is also used in further macro definitions.

In general, we expose the current world as a macro parameter $V$. This facilitates to identify proofs steps where axioms are not applied just with respect to the initially given current world but to some other reachable world.

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6. $ax_1^{\rightarrow}(V,P)$

---

Defined as

$$\mathsf{world}(V) \rightarrow (\mathsf{pos}(V,N') \rightarrow \neg\mathsf{pos}(V,P')),$$

where

$$\begin{aligned} N' &:= \ulcorner\neg P\urcorner,\\ P' &:= \ulcorner P\urcorner. \end{aligned}$$

The following macro renders Axiom 2 of Scott's version:

7. $ax_2(V, P, Q)$

Defined as

$$\begin{array}{ll} \mathsf{world}(V) & \rightarrow \\ (\mathsf{pos}(V, P') & \wedge \\ \;\forall W\ (\mathsf{r}(V, W) \rightarrow \forall X\ (\mathsf{e}(W, X) \rightarrow (P(W, X) \rightarrow Q(W, X)))) & \rightarrow \\ \;\mathsf{pos}(V, Q')), & \end{array}$$

where

$$\begin{array}{rcl} P' & := & \ulcorner P \urcorner, \\ Q' & := & \ulcorner Q \urcorner. \end{array}$$

► We can now derive the following lemma, called here Lemma 1 (it is not explicitly present in Scott's version), using just a single instance of each of $ax_1^{\rightarrow}$ and $ax_2$, where $\top$ and $\bot$ are the only predicates used for instantiating:

8. $lemma_1(V)$

Defined as

$$\mathsf{world}(V) \rightarrow \neg\mathsf{pos}(V, \ulcorner \bot \urcorner).$$

9. $pre\_lemma_1(V)$

Defined as

$$\begin{array}{ll} r\_world_1 & \wedge \\ topbot\_def & \wedge \\ topbot\_equiv\_equal & \wedge \\ ax_1^{\rightarrow}(V, \top) & \wedge \\ ax_2(V, \bot, \top). & \end{array}$$

This formula is valid: $pre\_lemma_1(\mathsf{v}) \rightarrow lemma_1(\mathsf{v})$.

Theorem 1 of Scott's version can be rendered as a macro with a predicate parameter:

10. $thm_1(V, P)$

Defined as

$$\begin{array}{ll} \mathsf{world}(V) & \rightarrow \\ (\mathsf{pos}(V, P') \rightarrow \exists W\ (\mathsf{r}(V, W) \wedge \exists X\ (\mathsf{e}(W, X) \wedge P(W, X)))), & \end{array}$$

where

$$P' := \ulcorner P \urcorner.$$

Instances of $\mathit{thm_1}(V, P)$ can be proven for arbitrary worlds $V$ and predicates $P$, from the respective instance of the axioms $\mathit{pre\_thm_1}(V, P)$. A further instance of $\mathit{ax_2}$ (beyond that used to prove $\mathit{lemma_1}$) is now required, with respect to $\bot$ and the given predicate $P$.

---

11. $\mathit{pre\_thm_1}(V, P)$

---

Defined as

$$\mathit{lemma_1}(V) \land \mathit{ax_2}(V, P, \bot).$$

This formula is valid: $\mathit{pre\_thm_1}(\mathsf{v}, \mathsf{p}) \to \mathit{thm_1}(\mathsf{v}, \mathsf{p})$.

When expanded, the formula $\mathit{pre\_thm_1}(\mathsf{v}, \mathsf{p}) \to \mathit{thm_1}(\mathsf{v}, \mathsf{p})$, whose validity has just been shown, looks as follows:

$$
\begin{array}{l}
(\mathsf{world}(\mathsf{v}) \to \neg\mathsf{pos}(\mathsf{v}, \ulcorner\bot\urcorner)) \quad \land \\
(\mathsf{world}(\mathsf{v}) \quad \to \\
\quad (\mathsf{pos}(\mathsf{v}, \ulcorner\mathsf{g}\urcorner) \land \\
\quad\quad \forall x\, (\mathsf{r}(\mathsf{v}, x) \to \forall y\, (\mathsf{e}(x, y) \to (\mathsf{g}(x, y) \to \bot(x, y)))) \quad \to \\
\quad\quad \mathsf{pos}(\mathsf{v}, \ulcorner\bot\urcorner))) \quad \to \\
(\mathsf{world}(\mathsf{v}) \quad \to \\
\quad (\mathsf{pos}(\mathsf{v}, \ulcorner\mathsf{g}\urcorner) \to \exists x\, (\mathsf{r}(\mathsf{v}, x) \land \exists y\, (\mathsf{e}(x, y) \land \mathsf{g}(x, y))))).
\end{array}
$$

# 3 Possibly God Exists

## 3.1 A Corollary of Theorem 1

Axiom 3 of Scott's version states that the predicate *god-like* has the property *is positive*. Together with Theorem 1 instantiated by *god-like* it is used to derive corollary *Coro*. This is rendered in the following formula definitions and validity statement, where *god-like* is represented by $\mathsf{g}$. Scott lets the definition of *god-like* precede Axiom 3. Since that definition is not required to prove *Coro*, we postpone its discussion to Section 4.1.

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12. $\mathit{ax_3}(V)$

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Defined as

$$\mathsf{world}(V) \to \mathsf{pos}(V, \ulcorner\mathsf{g}\urcorner).$$

13. $\mathit{coro}(V)$

Defined as

$$\mathsf{world}(V) \rightarrow \exists W\,(\mathsf{r}(V,W) \wedge \exists X\,(\mathsf{e}(W,X) \wedge \mathsf{g}(W,X))).$$

14. $\mathit{pre\_coro}(V)$

Defined as

$$\mathit{thm_1}(V,\mathsf{g}) \wedge \mathit{ax_3}(V).$$

This formula is valid: $\mathit{pre\_coro}(\mathsf{v}) \rightarrow \mathit{coro}(\mathsf{v})$.

# 4 Essence

## 4.1 Fragments of the Definition of God-Like

With macros $\mathit{def}_1^{\rightarrow}$ and $\mathit{def}_1^{\rightarrow\neg}$, defined now, we represent the left-to-right direction of the definition of *god-like* in Scott's version.

► Actually, only this direction is used in the proof of the existence of God.

The macros have a predicate as parameter that would be universally quantified in a higher-order version. The first macro expands into a formula in which the supplied predicate $P$ and its associated constant (see Section 2.2) do occur. In the second macro, their respective places are taken by the negated supplied predicate and the corresponding constant, that is, the constant associated with $\lambda vx.\neg P(v,x)$.

15. $\mathit{def}_1^{\rightarrow}(V,X,P)$

Defined as

$$\mathsf{g}(V,X) \rightarrow (\mathsf{pos}(V,P') \rightarrow P(V,X)),$$

where

$$P' := \ulcorner P \urcorner.$$

16. $\mathit{def}_1^{\rightarrow\neg}(V,X,P)$

Defined as

$$\mathsf{g}(V,X) \rightarrow (\mathsf{pos}(V,P') \rightarrow \neg P(V,X)),$$

where

$$P' := \ulcorner \neg P \urcorner.$$

## 4.2 The Essence of an Individual

The following macro *val_ess* renders the definiens of the *Ess*, or *essence of*, relationship between a predicate and an individual in Scott's version. It is originally a formula with predicate quantification, but without application of a predicate to a predicate. Our macro exposes the universally quantified predicate as parameter $Q$, which permits to use it just instantiated with some specific predicate. The quantified version can be still be expressed simply by prefixing a predicate quantifier upon $Q$.

---

17. $val\_ess(V, P, X, Q)$

---

Defined as

$$\begin{array}{ll} P(V, X) & \land \\ (Q(V, X) & \rightarrow \\ \ \forall W\ (\mathsf{r}(V, W) \rightarrow \forall Y\ (\mathsf{e}(W, Y) \rightarrow (P(W, Y) \rightarrow Q(W, Y))))). & \end{array}$$

► Eliminating the quantified predicate gives another view on *essence*:
Input: $\forall q\ val\_ess(\mathsf{v}, \mathsf{p}, \mathsf{x}, q)$.
Result of elimination:

$$\begin{array}{ll} \mathsf{p}(\mathsf{v}, \mathsf{x}) & \land \\ \forall y \forall z\ (\mathsf{e}(y, z) \land \mathsf{p}(y, z) \land \mathsf{r}(\mathsf{v}, y) \rightarrow y = \mathsf{v}) & \land \\ \forall y \forall z\ (\mathsf{e}(y, z) \land \mathsf{p}(y, z) \land \mathsf{r}(\mathsf{v}, y) \rightarrow z = \mathsf{x}). & \end{array}$$

We define a predicate ess in terms of the macro *val_ess*. This facilitates combining propositions that depend on the definiens with propositions that can be established independently from it:

---

18. $def\_ess(V, P)$

---

Defined as

$$\mathsf{world}(V) \rightarrow \forall X\ (\mathsf{ess}(V, P', X) \leftrightarrow \forall Q\ val\_ess(V, P, X, Q)),$$

where

$$P' := \ulcorner P \urcorner.$$

The following two observations are mentioned as *Note* in Scott's version. We express them with the predicate version ess of *Ess* to facilitate their use as axioms in other statements:

19. $note_1(V, P, Q)$

Defined as

$$\begin{array}{lr} \mathsf{world}(V) & \rightarrow \\ (\exists X\,(\mathsf{ess}(V, P', X) \land \mathsf{ess}(V, Q', X)) & \rightarrow \\ \forall W\,(\mathsf{r}(V, W) \rightarrow \forall Y\,(\mathsf{e}(W, Y) \rightarrow (P(W, Y) \leftrightarrow Q(W, Y))))), & \end{array}$$

where

$$\begin{aligned} P' &:= \ulcorner P \urcorner, \\ Q' &:= \ulcorner Q \urcorner. \end{aligned}$$

This formula is valid: $def\_ess(\mathsf{v}, \mathsf{p}_1) \land def\_ess(\mathsf{v}, \mathsf{p}_2) \rightarrow note_1(\mathsf{v}, \mathsf{p}_1, \mathsf{p}_2)$.

20. $note_2(V, P, X)$

Defined as

$$\begin{array}{lr} \mathsf{world}(V) & \rightarrow \\ (\mathsf{ess}(V, P', X) & \rightarrow \\ \forall W\,(\mathsf{r}(V, W) \rightarrow \forall Y\,(\mathsf{e}(W, Y) \rightarrow (P(W, Y) \rightarrow Y = X)))), & \end{array}$$

where

$$P' := \ulcorner P \urcorner.$$

This formula is valid: $def\_ess(\mathsf{v}, \mathsf{p}) \rightarrow note_2(\mathsf{v}, \mathsf{p}, \mathsf{x})$.

## 4.3 Deriving Theorem 2 – Almost

The right-to-left direction of Axiom 1 and Axiom 4 of Scott's version are rendered by macros $ax_1^{\leftarrow}$ and $ax_1^{\leftarrow}$, respectively, which are defined now. Both original axioms involve a universally quantified predicate that appears only in argument role. In the macro, that predicate appears simply as a parameter.

21. $ax_1^{\leftarrow}(V, P)$

Defined as

$$\mathsf{world}(V) \rightarrow (\neg\mathsf{pos}(V, P') \rightarrow \mathsf{pos}(V, N')),$$

where

$$\begin{aligned} N' &:= \ulcorner \neg P \urcorner, \\ P' &:= \ulcorner P \urcorner. \end{aligned}$$

22. $ax_4(V, P)$

Defined as

$$\mathsf{world}(V) \rightarrow (\mathsf{pos}(V, P') \rightarrow \forall W\, (\mathsf{r}(V, W) \rightarrow \mathsf{pos}(W, P'))),$$

where

$$P' := \ulcorner P \urcorner.$$

The following macro renders the rudiment of Theorem 2 of Scott's version. Originally, the $Q$ parameter is a universally quantified predicate inherited from the definiens of *Ess*.

23. $raw\_thm_2(V, X, Q)$

Defined as

$$\mathsf{world}(V) \rightarrow (\mathsf{e}(V, X) \rightarrow (\mathsf{g}(V, X) \rightarrow val\_ess(V, \mathsf{g}, X, Q))).$$

24. $pre\_thm_2(V, X, Q)$

Defined as

$$\begin{array}{ll} ax_1^{\leftarrow}(V, Q) & \wedge \\ \forall W\, (\mathsf{r}(V, W) \rightarrow \forall X\, (\mathsf{e}(W, X) \rightarrow def_1^{\rightarrow}(W, X, Q))) & \wedge \\ def_1^{\rightarrow\neg}(V, X, Q) & \wedge \\ ax_4(V, Q). & \end{array}$$

Theorem 2 would correspond to

$$\forall q \forall v \forall x\, raw\_thm_2(v, x, q).$$

The following statement can be proven for arbitrary individual symbols $\mathsf{v}, \mathsf{x}$ and predicate symbols $\mathsf{q}$. It is sufficient to derive a particular instance of the universally quantified Theorem 2 from a corresponding instance of the required axioms:

This formula is valid: $pre\_thm_2(\mathsf{v}, \mathsf{x}, \mathsf{q}) \rightarrow raw\_thm_2(\mathsf{v}, \mathsf{x}, \mathsf{q})$.

Moving a bit more to the full quantified version of Theorem 2, we can also prove:

25. $derive\_almost\_thm_2$

Defined as

$$\forall q\,(\forall v \forall x\,\exists q_q \exists not\_q_q\, pre\_thm_2(v,x,q) \quad \rightarrow \quad \forall v \forall x\, raw\_thm_2(v,x,q)).$$

This formula is valid: $derive\_almost\_thm_2$.

The following statement represents that Theorem 2 is implied by the required axioms $pre\_thm_2$, also under universal quantifications of its parameters and existential quantification of the predicate representatives:

26. $derive\_thm_2$

Defined as

$$\forall q\,\forall v \forall x\,\exists q_q \exists not\_q_q\, pre\_thm_2(v,x,q) \quad \rightarrow \quad \forall q\,\forall v \forall x\, raw\_thm_2(v,x,q).$$

► The validity of $derive\_thm_2$ seems derivable from the validity of $derive\_almost\_thm_2$ quite easily on a shallow level by Boolean reasoning and quantifier manipulation. The current version of *PIE*, however, would try to prove validity of $derive\_thm_2$ by eliminating the universal predicate quantifier in the antecedent, on which it does not succeed. Thus, at this point, with the current version of *PIE* there is a gap in the formal proof, which, however, should be resolvable in principle.

# 5 If God Exists, then Necessarily God Exists

## 5.1 Necessary Existence

The property *NE* in Scott's version applies to an individual and means that it necessarily exists if it has an essential property. The definiens of *NE* is rendered here by the macro $val\_ne$. It is originally expressed as a formula with predicate quantification, inherited from the definiens of *Ess* but without application of a predicate to a predicate.

27. $val\_ne(V,X)$

Defined as

$$\forall P\,(\forall Q\, val\_ess(V,P,X,Q) \quad \rightarrow \quad \forall W\,(\mathsf{r}(V,W) \rightarrow \exists Y\,(\mathsf{e}(W,Y) \land P(W,Y)))).$$

► Eliminating the quantified predicate gives another view on *NE*:

Input: $val\_ne(\mathsf{v}, \mathsf{x})$.
Result of elimination:

$$\forall y\,(\mathsf{r}(\mathsf{v}, y) \rightarrow y = \mathsf{v}) \land \forall y\,(\mathsf{r}(\mathsf{v}, y) \rightarrow \mathsf{e}(y, \mathsf{x})).$$

We define a predicate $\mathsf{ne}$ in terms of the macro $val\_ne$, in analogy to the definition of the predicate $\mathsf{ess}$:

---

28. $def\_ne(V, X)$

---

Defined as

$$\mathsf{world}(V) \rightarrow (\mathsf{e}(V, X) \rightarrow (\mathsf{ne}(V, X) \leftrightarrow val\_ne(V, X))).$$

## 5.2 Deriving that if God Exists, then Necessarily God Exists

The statement $\exists x\,\mathsf{g}(x) \rightarrow \Box\exists x\,\mathsf{g}(x)$ is used as an unlabelled lemma in Scott's version. In [BWW17, Fig. 1] it is called *L1*. We call it here *Lemma 2* and render it below in Macro 32 as $lemma_2$. In Scott's version it can be derived from Theorem 2, the definitions of *NE* and *Ess* as well as a further axiom, *Axiom 5*. Actually, the proof from these preconditions is largely independent from the definientia of *NE* and *Ess*. reconstruction. The following formula renders *Theorem 2*, now expressed in terms of the predicate $\mathsf{ess}$:

---

29. $thm_2(V, X)$

---

Defined as

$$\mathsf{world}(V) \rightarrow (\mathsf{e}(V, X) \rightarrow (\mathsf{g}(V, X) \rightarrow \mathsf{ess}(V, \ulcorner \mathsf{g} \urcorner, X))).$$

The following formula renders a fragment of the definition of *NE* on a "shallow" level, that is, in terms of just the predicates $\mathsf{ess}$ and $\mathsf{ne}$, without referring to $val\_ess$ and $val\_ne$:

---

30. $def_3^{\rightarrow}(V, X, P)$

---

Defined as

$$\begin{array}{l} \mathsf{world}(V) \rightarrow \\ (\mathsf{e}(V, X) \rightarrow \\ \;(\mathsf{ne}(V, X) \rightarrow \\ \;\;(\mathsf{ess}(V, P', X) \rightarrow \forall W\,(\mathsf{r}(V, W) \rightarrow \exists Y\,(\mathsf{e}(W, Y) \land P(W, Y)))))), \end{array}$$

where

$$P' \ := \ \ulcorner P \urcorner.$$

Correctness of $def_3^{\rightarrow}$ can be established by showing that it follows from definitions of $\mathsf{ess}$ and $\mathsf{ne}$ with definientia according to *val_ess* and *val_ne*:

This formula is valid: $def\_ess(\mathsf{v},\mathsf{g}) \land def\_ne(\mathsf{v},\mathsf{x}) \rightarrow def_3^{\rightarrow}(\mathsf{v},\mathsf{x},\mathsf{g})$.

(Validating this implication revealed a subtle shortcoming of the current version of *PIE*: If the biconditional signs in *def_ess* and *def_ess* would be replaced by implication signs, the implication just shown should also be valid. Although elimination on the involved predicate quantifiers should in principle succeed as it does in the variant with biconditionals, *PIE* currently seems to fail there.)

The remaining macros in this section render *Axiom 5* of Scott's version, the lemma $\exists x\, \mathsf{g}(x) \rightarrow \Box\exists x\, \mathsf{g}(x)$ mentioned above and preconditions for proving it.

---

31. $ax_5(V)$

---

Defined as

$$\mathsf{world}(V) \rightarrow \mathsf{pos}(V, \ulcorner \mathsf{ne} \urcorner).$$

---

32. $lemma_2(V)$

---

Defined as

$$\begin{array}{ll} \mathsf{world}(V) & \rightarrow \\ (\exists X\, (\mathsf{e}(V,X) \land \mathsf{g}(V,X)) & \rightarrow \\ \ \forall W\, (\mathsf{r}(V,W) \rightarrow \exists Y\, (\mathsf{e}(W,Y) \land \mathsf{g}(W,Y)))). & \end{array}$$

---

33. $pre\_lemma_2(V,X)$

---

Defined as

$$ax_5(V) \land def_1^{\rightarrow}(V,X,\mathsf{ne}) \land def_3^{\rightarrow}(V,X,\mathsf{g}) \land thm_2(V,X).$$

This formula is valid: $\forall v\, (\forall x\, pre\_lemma_2(v,x) \rightarrow lemma_2(v))$.

# 6 Necessarily God Exists

## 6.1 Proving the Main Result, Theorem 3

The following formula states *Theorem 3* of Scott's version, the overall result to show:

34. $thm_3(V)$

---

Defined as

$$\mathsf{world}(V) \rightarrow \forall W\, (\mathsf{r}(V, W) \rightarrow \exists Y\, (\mathsf{e}(W, Y) \wedge \mathsf{g}(W, Y))).$$

In proving *Theorem 3*, Scott proceeds from the lemma called here $lemma_2$ (Macro 32) and the corollary *Coro*, which corresponds to our Macro 13. He applies the modal axiom *E* (or *5*), which expresses that the accessibility relation is Euclidean. As shown apparently first in [BW14], *Theorem 3* can not be just proven in the modal logic S5, but also in KB, whose accessibility relation is less constrained. In particular, the modal axiom *B*, which expresses that the accessibility relation is symmetric, holds in KB. We show that the proof is possible for a Euclidean as well as a symmetric accessibility relation in a single statement by presupposing the disjunction of both properties:

---

35. *euclidean*

---

Defined as

$$\forall x \forall y \forall z\, (\mathsf{r}(x, y) \wedge \mathsf{r}(x, z) \rightarrow \mathsf{r}(z, y)).$$

---

36. *symmetric*

---

Defined as

$$\forall x \forall y\, (\mathsf{r}(x, y) \rightarrow \mathsf{r}(y, x)).$$

---

37. $pre\_thm_3(V)$

---

Defined as

$$r\_world_1 \wedge \forall v\; lemma_2(v) \wedge coro(V).$$

This formula is valid: $symmetric \vee euclidean \rightarrow (pre\_thm_3(\mathsf{v}) \rightarrow thm_3(\mathsf{v}))$.

Precondition $pre\_thm_3$ includes *coro* instantiated with just the current world and $lemma_2$ with a universal quantifier upon the world parameter. If fact, using $lemma_2$ there just instantiated with the current world would not be sufficient to derive $thm_3$:

This formula is not valid: $symmetric \vee euclidean \rightarrow (r\_world_1 \wedge lemma_2(\mathsf{v}) \wedge coro(\mathsf{v}) \rightarrow thm_3(\mathsf{v}))$.

# 7 Monotheism

In Fitting's system the proposition $\exists x \forall y\,(\mathsf{g}(y) \leftrightarrow y = x)$ can be derived [Fit02, Section 7.1]. This can be proven in our system from $\mathit{thm}_2$, $\mathit{note}_2$ and $\mathit{thm}_3$ under the additional assumption of reflexivity of the accessibility relation. Without that assumption, it can be shown that $\Box\exists x\Box\forall y\,(\mathsf{g}(y) \leftrightarrow y = x)$:

---

38. *pre_monotheism*

---

Defined as

$$\begin{array}{ll} \forall x \forall v\; \mathit{thm}_2(v, x) & \land \\ \forall x \forall v\; \mathit{note}_2(v, \mathsf{g}, x) & \land \\ \forall x \forall v\; \mathit{thm}_3(v) & \land \\ \mathit{r\_world}. & \end{array}$$

---

39. *monotheism*

---

Defined as

$$\forall v\, \exists x\, (\mathsf{e}(v, x) \land \forall y\, (\mathsf{e}(v, y) \rightarrow (\mathsf{g}(v, y) \leftrightarrow y = x))).$$

This formula is valid: $\mathit{pre\_monotheism} \land \mathit{reflexive} \rightarrow \mathit{monotheism}$.

---

40. *nec_monotheism*

---

Defined as

$$\begin{array}{l} \forall v \forall w\, (\mathsf{r}(v, w) \rightarrow \\ \quad \exists x\, (\mathsf{e}(w, x) \land \\ \quad\quad \forall w_1\, (\mathsf{r}(w, w_1) \rightarrow \forall y\, (\mathsf{e}(w_1, y) \rightarrow (\mathsf{g}(w_1, y) \leftrightarrow y = x))))). \end{array}$$

This formula is valid: $\mathit{pre\_monotheism} \rightarrow \mathit{nec\_monotheism}$.

# 8 Modal Collapse

A well-known objection to Gödel's theory is that it implies modal collapse [Sob87].

---

41. *collapse*

---

Defined as

$$\forall x \forall y\, (\mathsf{r}(x, y) \rightarrow y = x).$$

In our system, modal collapse can be derived from the following preconditions, selected after Fitting's reconstruction [Fit02, Chapter 11, Section 8] of Sobel's proof [Sob04, Sob87]:

---

42. *pre_collapse*

---

Defined as

$$\begin{array}{ll} \forall x \forall v\, \mathit{thm_2}(v, x) & \land \\ \forall x \forall v\, \mathit{thm_3}(v) & \land \\ \forall v\, \mathit{def\_ess}(v, \mathsf{g}) & \land \\ \mathit{r\_world} & \land \\ \mathit{reflexive}. & \end{array}$$

This formula is valid: $\mathit{pre\_collapse} \rightarrow \mathit{collapse}$.

In presence of *collapse*, the choice between frame conditions *symmetric* and *euclidean* (or the modal logics KB and S5) becomes immaterial, as both properties are implied by *collapse*. Also Axiom 4 is in presence of *collapse* redundant.

This formula is valid: $\mathit{collapse} \rightarrow \mathit{symmetric} \land \mathit{euclidean}$.
This formula is valid: $\mathit{collapse} \rightarrow \mathit{ax_4}(\mathsf{v}, \mathsf{p})$.

# 9 Alternate Weaker Preconditions for Lemma 1

The precondition $\mathit{pre\_lemma_1}$ used in Section 2 to derive $\mathit{lemma_1}$ includes

$$\mathit{topbot\_def} \land \mathit{topbot\_equiv\_equal}.$$

The following formula is a weaker formula that is also sufficient for deriving $\mathit{lemma_1}$:

---

43. $\mathit{topbot\_alt_1}$

---

Defined as

$$\begin{array}{ll} \forall v\, (\mathsf{world}(v) \rightarrow \forall x\, (\mathsf{e}(v, x) \rightarrow \top(v, x))) & \land \\ \forall v\, (\mathsf{world}(v) \rightarrow (\mathsf{pos}(v, \ulcorner \bot \urcorner) \rightarrow \mathsf{pos}(v, \ulcorner \neg \top \urcorner))). & \end{array}$$

This formula is valid: $\mathit{topbot\_def} \land \mathit{topbot\_equiv\_equal} \rightarrow \mathit{topbot\_alt_1}$.

---

44. $\mathit{pre\_lemma\_1\_drop\_topbot}(V)$

---

Defined as

$$F,$$

where

$$F \text{ is like } \mathit{pre\_lemma_1}(V) \text{ except}$$
$$\top \text{ instead of } \mathit{topbot\_def}$$
$$\top \text{ instead of } \mathit{topbot\_equiv\_equal}.$$

This formula is valid: $\mathit{topbot\_alt_1} \land \mathit{pre\_lemma\_1\_drop\_topbot}(\mathsf{v}) \rightarrow \mathit{lemma_1}(\mathsf{v})$.

A third possibility to derive $\mathit{pre\_lemma_1}$ is with the formula $\mathit{topbot\_alt_2}$ defined below, which is like $\mathit{topbot\_alt_1}$ except that $\top$ in the first conjunct is replaced by $\neg\bot$:

---

45. $\mathit{topbot\_alt_2}$

---

Defined as

$$F,$$

where

$$F \text{ is like } \mathit{topbot\_alt_1} \text{ except}$$
$$\neg\bot(V, X) \text{ instead of } \top(V, X).$$

This formula is valid: $\mathit{topbot\_def} \land \mathit{topbot\_equiv\_equal} \rightarrow \mathit{topbot\_alt_2}$.
This formula is valid: $\mathit{topbot\_alt_2} \land \mathit{pre\_lemma\_1\_drop\_topbot}(\mathsf{v}) \rightarrow \mathit{lemma_1}(\mathsf{v})$.

# 10 Frame Conditions for Deriving Theorem 3

## 10.1 The Weakest Sufficient Condition

We turn again to the proof of $\mathit{thm_3}$ (Macro 34) from $\mathit{pre\_thm_3}$ (Macro 37) in Section 6, where we used the additional frame condition $\mathit{euclidean} \lor \mathit{symmetric}$. The question is now, whether it is possible to find a weaker frame condition for deriving $\mathit{thm_3}$. Actually, the weakest such frame condition can be characterized as a second-order formula, the *weakest sufficient condition*, described briefly in Section 1.1, on the accessibility relation and possibly the domain membership relation:

---

46. $\mathit{wsc\_thm_3}$

---

Defined as

$$\forall g\, \forall v\, (\mathit{pre\_thm_3}(v) \rightarrow \mathit{thm_3}(v)).$$

However, elimination fails for this formula (at least with the current version of *PIE*). The idea is now to replace $\mathit{pre\_thm_3}$ with a weaker formula such that elimination becomes possible. We investigate this first in a simplified scenario.

## 10.2 Frame Conditions in a Modal Propositional Setting

The following macros specify versions of the formulas involved in deriving $\mathit{thm_3}$ from $\mathit{lemma_2}$ which are simplified in that they are just for propositional modal logics:

47. $\mathit{lemma_2\_simp}(V)$

Defined as

$$\mathsf{g}(V) \rightarrow \forall W\, (\mathsf{r}(V, W) \rightarrow \mathsf{g}(W)).$$

48. $\mathit{coro\_simp}(V)$

Defined as

$$\exists W\, (\mathsf{r}(V, W) \wedge \mathsf{g}(W)).$$

49. $\mathit{thm_3\_simp}(V)$

Defined as

$$\forall W\, (\mathsf{r}(V, W) \rightarrow \mathsf{g}(W)).$$

50. $\mathit{pre\_thm_3\_simp}(V)$

Defined as

$$\forall v\, \mathit{lemma_2\_simp}(v) \wedge \mathit{coro\_simp}(V).$$

However, elimination to obtain the weakest sufficient frame condition as a first-order formula still fails for the simplified scenario (at least with *PIE*). The corresponding second-order formula is $\forall g\, (\mathit{pre\_thm_3\_simp}(\mathsf{v}) \rightarrow \mathit{thm_3\_simp}(\mathsf{v}))$. The issue is now to find a weaker formula in which elimination succeed. We inspect the clausal tableau proof of the following task: This formula is valid: $\mathit{symmetric} \rightarrow (\mathit{pre\_thm_3\_simp}(\mathsf{v}) \rightarrow \mathit{thm_3\_simp}(\mathsf{v}))$.

It is shown in Figure 1. Actually only two instances of $\forall v\, \mathit{lemma_2\_simp}(v)$ are used in the proof. The following formula is a version of $\mathit{pre\_thm_3\_simp}$ with the required two instances, the second one inserted into an unfolding of $\mathit{coro\_simp}$:

51. $\mathit{pre\_thm_3\_simp\_inst}(V)$

Defined as

$$\mathit{lemma_2\_simp}(V) \wedge \exists W\, (\mathsf{r}(V, W) \wedge \mathsf{g}(W) \wedge \mathit{lemma_2\_simp}(W)).$$

The instantiated preconditions $\mathit{pre\_thm_3\_simp\_inst}$ are indeed implied by the original preconditions:

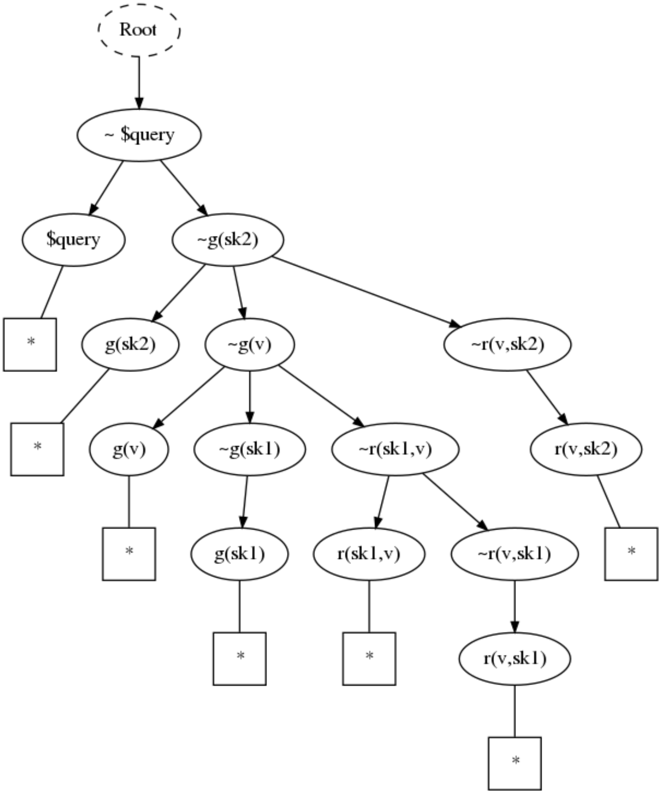


Figure 1: Clausal tableau proof – see discussion following Macro 50. The two instances of $\forall v \; lemma_2\_simp(v)$ appear in the clausal tableau as the two ternary clauses.

This formula is valid: $\mathit{pre\_thm_3\_simp}(\mathsf{v}) \rightarrow \mathit{pre\_thm_3\_simp\_inst}(\mathsf{v})$.

The instantiated preconditions *pre_thm₃_simp_inst* are sufficiently strong to derive $\mathit{thm_3\_simp}$, under the additional precondition *symmetric*, and, alternatively, also under the additional precondition *euclidean*:

This formula is valid: $\mathit{symmetric} \rightarrow (\mathit{pre\_thm_3\_simp\_inst}(\mathsf{v}) \rightarrow \mathit{thm_3\_simp}(\mathsf{v}))$.
This formula is valid: $\mathit{euclidean} \rightarrow (\mathit{pre\_thm_3\_simp\_inst}(\mathsf{v}) \rightarrow \mathit{thm_3\_simp}(\mathsf{v}))$.

► With $\mathit{pre\_thm_3\_simp\_inst}$ as precondition for $\mathit{thm_3\_simp}$ elimination to obtain the weakest sufficient frame condition as a first-order formula now succeeds:

Input: $\forall g\, \forall v\, (\mathit{pre\_thm_3\_simp\_inst}(v) \rightarrow \mathit{thm_3\_simp}(v))$.
Result of elimination:

$$\forall x \forall y \forall z\, (\mathsf{r}(x,y) \land \mathsf{r}(x,z) \rightarrow \mathsf{r}(y,x) \lor \mathsf{r}(y,z) \lor x = y \lor y = z).$$

We write the resulting first-order formula in a slightly different form and give it a name:

52. *frame_cond_simp*

Defined as

$$\forall x \forall y \forall z\, (\mathsf{r}(x,y) \land \mathsf{r}(x,z) \land y \neq x \land z \neq y \rightarrow \mathsf{r}(y,x) \lor \mathsf{r}(y,z)).$$

This formula is valid: $\mathit{frame\_cond\_simp} \leftrightarrow \mathit{last\_result}$.

► The obtained frame condition is under the assumption of reflexivity of the accessibility relation equivalent to $\mathit{symmetric} \lor \mathit{euclidean}$, and without that assumption strictly weaker:

53. *reflexive*

Defined as

$$\forall x\, \mathsf{r}(x,x).$$

This formula is valid: $\mathit{reflexive} \rightarrow (\mathit{symmetric} \lor \mathit{euclidean} \leftrightarrow \mathit{frame\_cond\_simp})$.
This formula is valid: $\mathit{symmetric} \lor \mathit{euclidean} \rightarrow \mathit{frame\_cond\_simp}$.
This formula is not valid: $\mathit{frame\_cond\_simp} \rightarrow \mathit{symmetric} \lor \mathit{euclidean}$.

► Thus we have shown for the propositional modal setting that the first-order formula *frame_cond_simp* is the weakest frame condition to derive $\mathit{thm_3\_simp}(\mathsf{v})$ from $\mathit{pre\_thm_3\_simp\_inst}(\mathsf{v})$. Under the assumption of reflexivity this condition is equivalent to $\mathit{symmetric} \lor \mathit{euclidean}$. Without that assumption it is strictly weaker. As a corollary it follows that this condition is also sufficient as frame condition to derive $\mathit{thm_3\_simp}(\mathsf{v})$ from $\mathit{pre\_thm_3\_simp}(\mathsf{v})$, but in in this case it is not necessarily the weakest such frame condition.

► The pattern in which we proceeded here might possibly be also applicable in other situations. It can be described as follows: Our original problem involved a universal second-order quantifier, for which elimination fails. We considered a stronger universal second-order formula on which elimination succeeds. Since $\forall$ can be represented by $\neg\exists\neg$, with respect to *existential* predicate quantification, this corresponds to considering a *weaker* second-order formula. We obtained a solution of the modified problem that also provides a solution of the original problem, although not necessarily the "best" solution (*weakest* sufficient condition, in our case).

## 10.3 Considering Modal Predicate Logic Again

We now turn back to the problem of finding weak frame conditions for

$$\mathit{pre\_thm_3}(\mathsf{v}) \rightarrow \mathit{thm_3}(\mathsf{v}).$$

► In fact, the frame condition obtained for the propositional case also works in this case:

This formula is valid: $\mathit{frame\_cond\_simp} \rightarrow (\mathit{pre\_thm_3}(\mathsf{v}) \rightarrow \mathit{thm_3}(\mathsf{v}))$.

Can further results be obtained for the modal predicate logic case? Our first attempt is to form $\mathit{pre\_thm_3\_inst}$ in analogy to $\mathit{pre\_thm_3\_simp\_inst}$:

---

54. $\mathit{pre\_thm_3\_inst}(V)$

---

Defined as

$$\begin{array}{ll} \mathit{r\_world_1} & \wedge \\ \mathit{lemma_2}(V) & \wedge \\ (\mathsf{world}(V) \rightarrow \exists W\, (\mathsf{r}(V,W) \wedge \exists X\, (\mathsf{e}(W,X) \wedge \mathsf{g}(W,X)) \wedge \mathit{lemma_2}(W))). & \end{array}$$

Unfortunately, however, elimination on

$$\forall g\, \forall v\, (\mathit{pre\_thm_3\_inst}(v) \rightarrow \mathit{thm_3}(v))$$

does not succeed with *PIE*. We thus build a formula with a more tight integration of $\mathit{lemma_2}$ and *coro*:

---

55. $\mathit{pre\_thm_3\_tight}(V)$

---

Defined as

$$\begin{array}{ll} \mathit{lemma_2}(V) & \wedge \\ (\mathsf{world}(V) & \rightarrow \\ \quad \exists W\, (\mathsf{r}(V,W) & \wedge \\ \qquad \exists X\, (\mathsf{e}(W,X) \wedge \mathsf{g}(W,X)) & \wedge \\ \qquad \forall W_1\, (\mathsf{r}(W,W_1) \rightarrow \exists Y\, (\mathsf{e}(W_1,Y) \wedge \mathsf{g}(W_1,Y))))). & \end{array}$$

It satisfies our basic requirements:

This formula is valid: $\mathit{pre\_thm_3}(\mathsf{v}) \rightarrow \mathit{pre\_thm_3\_tight}(\mathsf{v})$.
This formula is valid: $\mathit{frame\_cond\_simp} \rightarrow (\mathit{pre\_thm_3\_tight}(\mathsf{v}) \rightarrow \mathit{thm_3}(\mathsf{v}))$.

And, it permits elimination. Since the result formula obtained from the elimination procedure looks clumsy, here it is simplified "by hand" and mechanically verified:

---

56. *frame_cond_tight*

---

Defined as

$$\begin{array}{l}\forall v\,(\mathsf{world}(v) \rightarrow \\ \quad \forall w\,(\mathsf{r}(v,w) \land w \neq v \land \exists x\,\mathsf{e}(w,x) \rightarrow \\ \quad\quad \forall w_1\,(\mathsf{r}(v,w_1) \land w \neq w_1 \rightarrow \\ \quad\quad\quad \exists w_2\,(\mathsf{r}(w,w_2) \land (\exists v\,\mathsf{e}(w_2,v) \rightarrow w_1 = w_2 \lor v = w_2))))).\end{array}$$

This formula is valid: $\mathit{frame\_cond\_tight} \leftrightarrow \forall g\,\forall v\,(\mathit{pre\_thm_3\_tight}(v) \rightarrow \mathit{thm_3}(v))$.

Under the preconditions $\mathit{r\_world}$, and *nonempty*, which expresses that all worlds have a nonempty domain, the frame condition *frame_cond_tight* is equivalent to the frame condition *frame_cond_simp* (Macro 52) of the simplified scenario:

---

57. *nonempty*

---

Defined as

$$\forall v\,(\mathsf{world}(v) \rightarrow \exists x\,\mathsf{e}(v,x)).$$

This formula is valid: $\mathit{r\_world} \land \mathit{nonempty} \rightarrow (\mathit{frame\_cond\_tight} \leftrightarrow \mathit{frame\_cond\_simp})$.

# 11 Auxiliary Definitions

---

58. *last_result*

---

Defined as

$$X,$$

where

last_ppl_result(X).

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