1. Craig’s Interpolation Theorem [Cra57]

Definition. Let $F$ and $G$ be first-order formulas such that $F \vdash G$. A Craig-Lyndon interpolant of $F$ and $G$ is a first-order formula $H$ such that

1. $F \vdash H \vdash G$.
2. A predicate occurs positively (negatively) in $H$ only if it occurs positively (negatively) in both $F$ and $G$.
3. A function occurs in $H$ only if it occurs in both $F$ and $G$.

$p \leq q \rightarrow p \leq q \rightarrow H$.

2. Definitions Compositor as Interpolation

The following statements are equivalent:

1. $F \vdash H$.
2. $H$ is a compositor of $F$.

Rationale: $F \vdash p(x) \rightarrow p(x)$ if and only if $F \vdash p(x) \rightarrow p(x)$.

3. Construction of First-Order Craig Interpolants

Basic approach: extract $H$ from a proof of $F \vdash G$.

4. Clausal Tableaux [Let99a] (aka Clause Tableaux [Häh01])

Definition. A clausal tableau for a formula $F$ is a finite ordered tree whose nodes $N$ with exception of the root are labeled with a first-order literal, denoted by $\lnot i(t)$, such that: For each node $N$ the disjunction of the labels $\{\lnot i(t) \mid v \in N\}$ in their left-to-right order, denoted by clause($N$), is an instance of a clause in $F$.

A node $N$ is called closed if and only if it has an ancestor $N'$ with clause($N') = \{\lnot i(t)\}$. With a closed node $N$, a particular such ancestor $N'$ is associated with a target($N$). A tableau is called closed if and only if all of its leaves are closed.

The universal closure of a clausal formula $F$ (with at least one constant) is unsatisfiable if and only if there exists a clausal tableau for $F$. If and only if there exists a closed clausal tableau for a formula $F'$, the terms formed from functions in $F$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

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$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

5. Interpolant Extraction from Clausal Ground Tableaux

Definition. A colored clausal tableau for $F$ and $\text{color}(N)$ is a clausal tableau for $\text{color}(N)$ whose nodes $N$ with exception of the root are labeled additionally with $\text{color}(N) \in \{\text{red}, \text{blue}\}$ such that if $N$ is a leaf of $N$, then $\text{color}(N)$ is an instance of a clause in $F$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

Underlying Property. For all nodes $N$ of $\text{color}(N)$ is a Craig-Lyndon interpolant of $\text{branch}_{\text{color}(N)}(N)$ and $\text{branch}_{\text{color}(N)}(N)$, where $\text{branch}_{\text{color}(N)}(N)$ is the conjunct of the literal labels of Color nodes on the branch to $N$ including $N$.

10. Interpolant Construction for RQFO Formulas

Definition. Let $F$ and $G$ be RQFO formulas such that $F \vdash G$. Let $T$ be a clausal ground tableau for two clausal formulas obtained by classifying $\text{DEF}^*(F)$ and $\text{DEF}^*(G)$. Assume that $T$ is closed, eager, red-left for the set of all $\text{DEF}^*(F)$, $\text{DEF}^*(G)$, $\text{DEF}^*(F)$, $\text{DEF}^*(G)$, and $\text{DEF}^*(F)$, $\text{DEF}^*(G)$. For inner nodes $N$ of $T$ with children $N_1, N_2, \ldots, N_N$ define $\text{acc}(N)$, depending on the form 1–N of which clause($N$) is an instance:

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

Example: Interpolant from Clausal Ground Tableaux

With a closed node $N$ that originates in the node corresponding to $\text{leaf}(N)$, $\text{branch}_{\text{leaf}(N)}(N)$ is defined inductively for all nodes $N$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

$p(x) \land q(x) \land r(x) \rightarrow p(x)$.

Answer: $\text{branch}_{\text{leaf}(N)}(N)$.