

# **Second-Order Quantifier Elimination on Relational Monadic Formulas – A Basic Method and Some Less Expected Applications**

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## Second-Order Quantifier Elimination

- A computational task

Input: A **second-order** formula

$$\exists p (\forall x qx \rightarrow px) \wedge (\forall x px \rightarrow rx)$$

Output: An **equivalent first-order** formula

$$\forall x qx \rightarrow rx$$

- Related notions: uniform interpolation, forgetting, projection
- Applications
  - Computing frame correspondence properties of modal formulas
  - Computing circumscription and abductive explanations
  - Ontology reuse, ontology analysis, information hiding

## Second-Order Quantifier Elimination: A Look Back into History

- Boole ~1854, Peirce ~1880, Schröder ~1891: Algebra of Logic
- Löwenheim 1915, Skolem 1919: Deciding monadic formulas
- Behmann 1922  
*Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem*
  - Explicit formulation of the decision problem
  - [Rewriting based decision/elimination method for monadic formulas](#)
- Ackermann 1935  
*Untersuchungen über das Eliminationsproblem der mathematischen Logik*
  - [Ackermann's Lemma, Ackermann's Quantifier Switching](#)
  - Resolution-based elimination method
- Craig ~1960
- Sahlqvist 1975, van Benthem 1983  
Computing modal first-order frame correspondence properties
- Gabbay/Ohlbach 1991  
Resolution-based elimination method (SCAN)
- Szalas/Doherty/Łukaszewicz 1992  
[Elimination by rewriting to Ackermann's Lemma \(DLS, Ackermann approach\)](#)
- Gabbay/Schmidt/Szalas 2008: Monograph
- Since ~2006: [Forgetting in description logics](#)

1. Introduction
- 2. Behmann's Elimination Method**
3. Useful Second-Order Properties
4. Hidden Monadicity in Description Logics
5. The Ackermann Approach in View of Monadic Techniques
6. Conclusion

## Formula Classes Considered by Behmann

**MON**      The class of **relational monadic formulas** (Löwenheim class)

i.e. first-order formulas

- with only nullary and **unary predicates**
- **without functions** except constants

**MON=**      MON with equality

**QMON**      MON with predicate quantification

**QMON=**      MON with predicate quantification and equality

- All these classes are **decidable**
- **Second-order quantifier elimination** on a **QMON=** formula always succeeds, resulting in a **MON=** formula
- **Elimination** on a **QMON** formula might result in a **MON=** formula

## Behmann's Core Method and its Applications

- Behmann's core method eliminates  $\exists p$  in  $\exists p F$  where  $F$  is  $\text{MON}_=$
- It can be applied to **eliminate second-order quantifiers in  $\text{QMON}_=$**  formulas:
  - Repeatedly eliminate innermost second-order quantifiers
- It can be applied to **decide  $\text{QMON}_=$**  formulas:
  - Eliminate all second-order quantifiers in  $\exists p_1 \dots \exists p_n \exists x_1 \dots \exists x_m \exists c_1 \dots \exists c_k F$
  - The output is a  $\text{MON}_=$  sentence without predicates and constants

It is either true or false for all domain cardinalities with exception of a finite number

For all infinite domains the value is the same

## Outline of Behmann's Elimination Method

Operates by **equivalence preserving formula rewriting**

1. Normalization to “**innex form**” w.r.t. first-order quantifiers
  - Propagate quantifiers inwards such that their **scopes do not overlap**
  - This is possible for MON and with **counting quantifiers** for  $\text{MON}_=$
  - Requires **expensive rewritings** such as distribution among  $\wedge$  and  $\vee$
  - Stronger than **antiprenexing**, e.g. [Egly, 1994]
2. Normalization such that **subformulas**  $\exists p \dots$  are in “Eliminationshauptform”
  - Involves potentially expensive inward propagation of  $\exists p$
3. Conversion to match the left side of the **Basic Elimination Lemma**

### Proposition: Basic Elimination Lemma

$$\exists p (\forall x Fx \vee px) \wedge (\forall x Gx \vee \neg px) \equiv \forall x Fx \vee Gx$$

4. **Elimination by applying the Basic Elimination Lemma**
5. Postprocessing

## Other Decision Methods for Monadic Relational Formulas

- Innexing by a [generalization of Shannon's expansion](#) [Quine, 1945]
- Variants of [resolution](#) and [superposition](#) decide MON and MON<sub>=</sub> [Fermüller et al., 2001, Bachmair et al., 1993]
- Complexity considerations [Lewis, 1980, Börger et al., 1997]
  - A satisfiable MON formula has a model whose [cardinality is at most  \$2^{\text{number of predicates}}\$](#)
  - A model can be verified by a variant of innexing with DNF/CNF transformations where [only atoms present in the input are involved](#)  
The size never grows beyond a single exponential in the input size
  - Deciding satisfiability for MON and MON<sub>=</sub> is [NEXPTIME-complete](#)



1. Introduction
2. Behmann's Elimination Method
- 3. Useful Second-Order Properties**
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## Elimination and Introduction of Auxiliary Definitions, Ackermann's Lemma

- It is common practice to obtain small equi-satisfiable CNFs by introducing **auxiliary definitions** – e.g. Tseitin's encoding
- Second-order quantification allows to understand the introduction and elimination of such definitions as **equivalence preserving** transformations

### Proposition: Elimination and Introduction of Definitions

$$\exists p (\forall x px \leftrightarrow Gx) \wedge F[pt_1, \dots, pt_n] \equiv F[Gt_1, \dots, Gt_n]$$

### Proposition: Ackermann's Lemma [Ackermann, 1935a]

If all  $n$  indicated subformula occurrences in  $F[pt_1, \dots, pt_n]$ :  
(or, equivalently, in  $F[Gt_1, \dots, Gt_n]$ ) have the **same polarity**  $POL$ , then

$$\begin{aligned} \exists p (\forall x px \rightarrow Gx) \wedge F[pt_1, \dots, pt_n] &\equiv F[Gt_1, \dots, Gt_n], & \text{if } POL = \text{pos} \\ \exists p (\forall x px \leftarrow Gx) \wedge F[pt_1, \dots, pt_n] &\equiv F[Gt_1, \dots, Gt_n], & \text{if } POL = \text{neg} \end{aligned}$$

## Ackermann's Quantifier Switching

**Proposition: Ackermann's Quantifier Switching [Ackermann, 1935b]**

$$\exists p \forall x F[pxt_1, \dots, pxt_n] \equiv \forall x \exists p' F[p't_1, \dots, p't_n]$$

- [Ackermann, 1935b] applies it to **avoid Skolemization** and to allow **monadic techniques** or **Ackermann's Lemma**
- [van Benthem, 1983] applies it right-to-left to achieve prenex form w.r.t. second-order quantifiers

1. Introduction
2. Behmann's Elimination Method
3. Useful Second-Order Properties
- 4. Hidden Monadicity in Description Logics**
5. The Ackermann Approach in View of Monadic Techniques
6. Conclusion

## Normalized Representation of Description Logic KBs in Classical Logics

- A description logic (DL) **knowledge base** (KB) is a pair  $\langle TBox, ABox \rangle$
- Many DLs can be considered as **fragments of first-order logic**
- A KB in such a DL can be **translated to a first-order sentence**, based on the **standard translation** of modal logics
- This representation can be **normalized to a second-order formula** where
  - **Quantified auxiliary concept predicates** “define” subformulas
  - The matrix is a **generalized CNF**, in place of literals are **basic formulas**:

<i>Basic formula</i>	<i>Inducing DL construct</i>
$cx$	atomic concept, ABox assertion
$\neg cx$	atomic concept
$\exists y rxy \wedge dy$	qualified existential restriction
$\forall y \neg rxy \vee dy$	qualified value restriction
$x \neq a$	ABox assertion
$rx a$	ABox assertion

- Observe: All occurrences of **role predicates** have  $x$  as first argument

## Normalized Representation of Description Logic KBs: Example

$$KB = \langle \{ C_1 \sqsubseteq \exists R.(C_2 \sqcup C_3), \\ C_2 \sqsubseteq C_4 \}, \\ \{ C_1(A) \} \rangle$$

$$\forall x \text{ST}_x(KB) = \forall x \quad \begin{array}{l} c_1x \rightarrow \exists y rxy \wedge (c_2y \vee c_3y) \wedge \\ c_2x \rightarrow c_4x \wedge \\ c_1a \end{array}$$

$$\exists d_1 \forall x F = \exists d_1 \forall x \quad \begin{array}{l} c_1x \rightarrow \exists y rxy \wedge d_1y \wedge \\ d_1x \rightarrow c_2x \vee c_3x \wedge \\ c_2x \rightarrow c_4x \wedge \\ x = a \rightarrow c_1x \end{array}$$

## Satisfiability of $\mathcal{ALC}$ as $\text{QMON}_=$ Satisfiability

- The normalized translation

$$\exists d_1 \dots \exists d_k \forall x F$$

is equi-satisfiable with the closure of its free concept and role predicates:

$$\exists c_1 \dots \exists c_n \exists r_1 \dots \exists r_m \exists d_1 \dots \exists d_k \forall x F$$

- The predicate quantifiers can be reordered:

$$\exists c_1 \dots \exists c_n \exists d_1 \dots \exists d_k \exists r_1 \dots \exists r_m \forall x F$$

- Ackermann's Quantifier Switching** can be applied to the role predicates:

$$\exists c_1 \dots \exists c_n \exists d_1 \dots \exists d_k \forall x \exists r'_1 \dots \exists r'_m F[r_i x t \mapsto r'_i t]$$

- This is a  $\text{QMON}_=$  formula
- This technique applies to  $\mathcal{ALCOQH}$ : atomic concepts,  $\top$ ,  $\perp$ ,  $\neg$ ,  $\sqcap$ ,  $\exists R.C$ ,  $\forall R.C$  + nominals, qualified number restrictions, subroles

**Satisfiability of  $\mathcal{ALCOQH}$  KBs can be expressed as  $\text{QMON}_=$  satisfiability**

## Elimination for DLs as QMON= Elimination (I)

- The objective is to **apply second-order quantifier elimination** to

$$\exists p_1 \dots \exists p_n \forall x \text{ST}_x(KB),$$

where the  $p_i$  are concept predicates

- **Normalization** and further conversion yields an equivalent formula

$$S \wedge \exists c_1 \dots \exists c_l \exists d_1 \dots \exists d_k \forall x F,$$

where

- $S$  is a sentence with only role predicates
- $c_i$  are concept predicates to eliminate that only occur with argument  $x$ :
  - the  $p_i$
  - some of the auxiliary concept predicates
- $d_i$  are the remaining auxiliary concept predicates
- $F$  is a generalized CNF



## Elimination for DLs as QMON<sub>=</sub> Elimination (II)

- We introduce unary role predicates  $r'_i$  and store the original  $r_i$  in definitions:

Let  $Rx$  be  $\bigwedge_{1 \leq i \leq m} (\forall y r'_i y \leftrightarrow r_i x y)$ . Then our formula

$$S \wedge \exists c_1 \dots \exists c_l \exists d_1 \dots \exists d_k \forall x F$$

is equivalent to

$$S \wedge \exists c_1 \dots \exists c_l \exists d_1 \dots \exists d_k \forall x \exists r'_1 \dots \exists r'_m Rx \wedge F[r_i x t \mapsto r'_i t]$$

- We apply Ackermann's Quantifier Switching to the  $c_i$ :

$$S \wedge \exists d_1 \dots \exists d_k \forall x \exists r'_1 \dots \exists r'_m Rx \wedge \exists c'_1 \dots \exists c'_l F[r_i x t \mapsto r'_i t, c_i x \mapsto c'_i]$$

- $F[\dots]$  is now a MON<sub>=</sub> formula, thus we can eliminate the  $\exists c'_i$
- We then restore the binary role predicates according to  $Rx$  and delete  $Rx$
- Our formula then has the form

$$S \wedge \exists d_1 \dots \exists d_k \forall x F'$$

and we are finished if there are no  $d_i$

- This is the case if the DL has only limited (unqualified) restriction  
e.g.  $\exists R.\top$

## Elimination for DLs as QMON= Elimination (III)

**Concept elimination on KBs in *ALCOQH* with only limited restriction but in addition inverse roles can be performed by second-order quantifier elimination on QMON= and definition elimination**

- With inverse roles but only limited restriction, this covers typical members of the [DL-Lite](#) family

## Related Approaches in Description Logics

- **Two-phase forgetting for expressive DLs**  
[Koopmann and Schmidt, 2013b, Koopmann and Schmidt, 2015]
  - Based on resolution and a generalized Ackermann's Lemma
  - The phases are roughly related to the  $c_i$  and  $d_i$
  - Normalization is preserved throughout to allow re-translation to DL
- **Equi-satisfiable translations of DL-Lite into the one-variable fragment**  
[Artale et al., 2009]
- Specialized methods for **forgetting in DL-Lite**  
[Kontchakov et al., 2010, Wang et al., 2010]

1. Introduction
2. Behmann's Elimination Method
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- 5. The Ackermann Approach in View of Monadic Techniques**
6. Conclusion

## The Ackermann Approach to Second-Order Quantifier Elimination

- Also called **direct approach**
- Basic idea: **Rewrite** such that elimination can be performed by applying **Ackermann's Lemma** to subformulas
- Exemplified by **DLS**  
[Szałas, 1993, Doherty et al., 1997, Gustafsson, 1996, Conradie, 2006]
  1. **Preprocessing to a certain normal form**  
**This might fail**
  2. **Preparation for Ackermann's Lemma**  
Skolem functions might be introduced
  3. **Application of Ackermann's Lemma and un-Skolemization**  
**Un-Skolemization might fail**
  4. **Simplification**

## A Limitation of DLS – Obvious in View of Behmann's Method

- The preprocessing of DLS fails on some monadic inputs that, however, could be converted to the target normal form

The reason is that DLS first converts to negation normal form and then includes distribution of  $\wedge$  over  $\vee$ , but not distribution of  $\vee$  over  $\wedge$

## Ackermann's Lemma for "Semi Monadic" Formulas

### 2. Preparation for Ackermann's Lemma

Convert  $\exists p A \wedge B$ , where  $A$  ( $B$ ) contains  $p$  just **positively** (**negatively**), to

$$\exists f_1 \dots \exists f_k \exists p (\forall x A'x \rightarrow px) \wedge B'$$

### 3. Application of Ackermann's Lemma and un-Skolemization

Un-Skolemization might fail

**If  $A$  or  $B$  is a  $\text{MON}_=$  formula, the preparation for Ackermann's Lemma can be performed without introduction of Skolem functions**

- The result may contain several subformulas  $\exists p \dots$  that all match Ackermann's Lemma
- Elimination on **Sahlqvist formulas** matches the "semi monadic" case
  - Success on computing correspondence properties of **Sahlqvist formulas** is an investigated **completeness property of elimination methods** [Goranko et al., 2004, Conradie, 2006, Schmidt, 2012]
  - The core step of a special elimination method for Sahlqvist formulas (the **Sahlqvist-van Benthem substitution algorithm**) can be considered as application of **Ackermann's Lemma** on a "semi monadic" input

1. Introduction
2. Behmann's Elimination Method
3. Useful Second-Order Properties
4. Hidden Monadicity in Description Logics
5. The Ackermann Approach in View of Monadic Techniques
- 6. Conclusion**



## Open and Future Issues

- **Further investigation of methods for relational monadic formulas**
  - Complexity of QMON, QMON\_?
  - How can particular formula patterns occurring in applications be utilized?
  - What implementation techniques are relevant?
- **Further investigation of applications**
  - Taking finite relations and constraint databases into account
  - Deeper investigation of forgetting in DLs
  - First-order consequences vs. DL consequences
- **Follow-up issues on elimination**
  - Further completeness criteria of elimination methods
  - Further improvements of DLS
  - Investigation of Ackermann's resolution-based method and related works

## Summary

- **Restoration of Behmann's second-order quantifier elimination method**
  - Continued work by Schröder, inspired Ackermann
  - Developed along the notion of Entscheidungsproblem
  - Explicitly targeted at practical application – Computational Logic
  - Operates by equivalence preserving formula rewriting to normal forms
- **Observing aspects of Behmann's method that seem not yet digested**
  - Similarity to modern methods of the Ackermann approach
  - Beyond antiprenexing: innexing for the price of expensive operations
  - Completeness for monadic formulas as criterion for elimination methods
- **Expressing applications as elimination on relational monadic formulas**
  - Deciding *ALC* knowledge bases
  - Concept elimination in DL-Lite knowledge bases
  - Eliminability in “semi monadic” formulas, for example Sahlqvist formulas
- **Use of a methodology with equivalence preserving transformations**
  - First-order based, applying second-order properties like Ackermann's Quantifier Switching and Ackermann's Lemma

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