Learning from Łukasiewicz and Meredith: Investigations into Proof Structures

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Learning from Łukasiewicz and Meredith: Investigations into Proof Structures

- We will look in depth into a historic formal **human** proof
  - Completeness of **Jan Łukasiewicz's shortest single axiom for IF**, the implicational fragment of propositional logic ($\rightarrow$ as only logic operator) within a first-order metatheory
  - Proof by Łukasiewicz published in 1948 with the method of substitution and detachment, refined ca. 1963 by **Carew A. Meredith**
  - Modern ATP systems produce much longer proofs
  - Our interest links **improving a given proof** and **improving proof search**
- Issues to **learn** from Łukasiewicz and Meredith
  - What techniques are behind their success?
  - Are these useful in modern ATP?
- We aim at providing a simple **basis for machine learning** in first-order ATP
  - Global features of proofs instead of local orientedness of calculi
  - Human discovery of relevant special techniques as model for machine learning

History of Łukasiewicz's Single Axiom for the Implicational Fragment

1925	*Tarski*: Can IF be axiomatized by a single axiom? Solved, but with long axioms
1925	*Łukasiewicz*: Are there shorter axioms? Found size 25
1926–32	*Wajsberg*, *Łukasiewicz*: sizes 25, 17
1936	*Łukasiewicz* – the shortest: size 13
1948	*Łukasiewicz*: publication of the shortest formal completeness proof by substitution and detachment
1954	*Church*: general theory of the matter?
1956	*Meredith*: Condensed Detachment (CD) – unification anticipated
1963	*Meredith*: refined version of Łukasiewicz's proof
1968–70	*Tursman*, *Thomas*: uniqueness of the single axiom
1983	*Kalman*: CD as a rule of inference formalizations of CD, meta-level formulation for (hyper-)resolution
1988	*Pfenning* @ CADE: brings Łukasiewicz's single axiom to attention meta-level formulation for first-order ATP, experiments with PTTP
1990	*Hindley*, *D. Meredith*: principal type schemes and CD
1992	*McCune*, *Wos* @ CADE: Experiments in autom. ded. with CD Ł's single axiom: first truly difficult CD problem solved by Otter
1993–	TPTP: 205 CD problems in LCL; 8 rated 1; 49 rated >0 (2021)
2001	*Wos*, *Ulrich*, *Veroff*, *Harris*, *Fitelson*: JAR special issue on CD

The Proven Problem: Completeness of Łukasiewicz's Single Axiom for IF

Involved Formulas

	Nickname	Łukasiewicz's syntax	First-order representation
Tarski-Bernays Axioms	*Simp*	$CpCqp$	$\forall pq\, \mathsf{P}(\mathsf{i}(p, \mathsf{i}qp))$
	Peirce	$CCCpqpp$	$\forall pq\, \mathsf{P}(\mathsf{i}(\mathsf{i}(\mathsf{i}pq), p), p)$
	Syll	$CCpqCCqrCpr$	$\forall pqr\, \mathsf{P}(\mathsf{i}(\mathsf{i}pq, \mathsf{i}(\mathsf{i}qr, \mathsf{i}pr)))$
	Łukasiewicz	$CCCpqrCCrpCsp$	$\forall pqrs\, \mathsf{P}(\mathsf{i}(\mathsf{i}(\mathsf{i}pq, r), \mathsf{i}(\mathsf{i}rp, \mathsf{i}sp)))$

Detachment Meta-Axiom

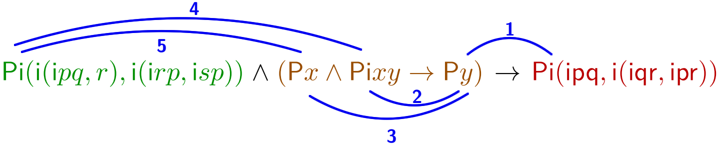
$$\textit{Detachment} \overset{\text{def}}{=} \forall xy\, (\mathsf{P}x \land \mathsf{P}\mathsf{i}xy \rightarrow \mathsf{P}y)$$

minor premise ($\mathsf{P}x$)
major premise ($\mathsf{P}\mathsf{i}xy$)
conclusion ($\mathsf{P}y$)

Most Difficult Considered Subproblem

$$\models \textit{Łukasiewicz} \land \textit{Detachment} \rightarrow \textit{Syll}$$

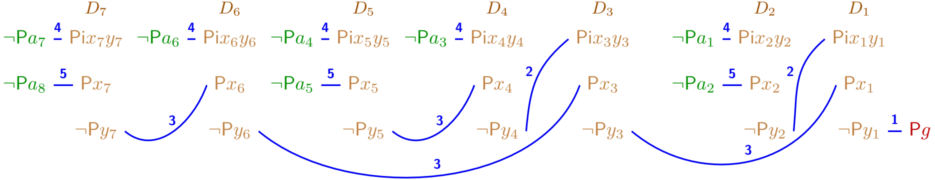
Connection Method Representation of the Subproblem



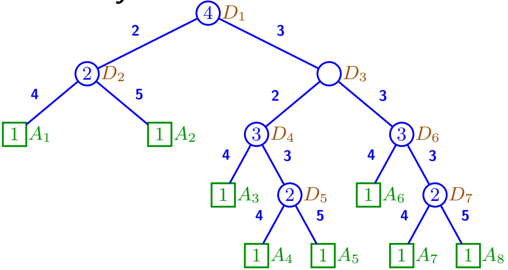
$$\mathsf{P}\mathsf{i}(\mathsf{i}(\mathsf{i}pq, r), \mathsf{i}(\mathsf{i}rp, \mathsf{i}sp)) \land (\mathsf{P}x \land \mathsf{P}\mathsf{i}xy \rightarrow \mathsf{P}y) \rightarrow \mathsf{P}\mathsf{i}(\mathsf{ipq}, \mathsf{i}(\mathsf{iqr}, \mathsf{ipr}))$$

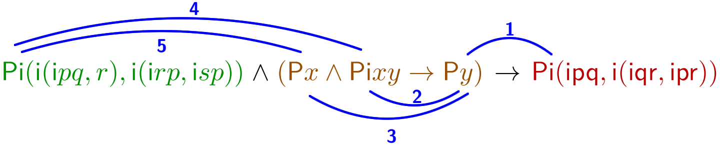
A Proof in Different Representations

Connection Method Proof Structure



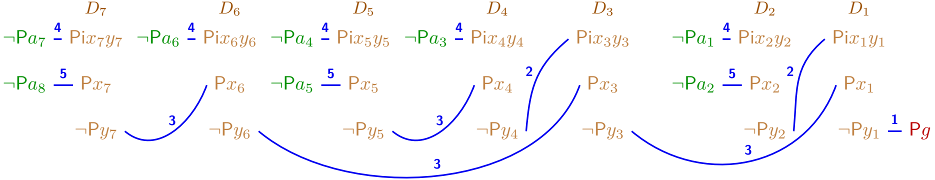
Full Binary Tree



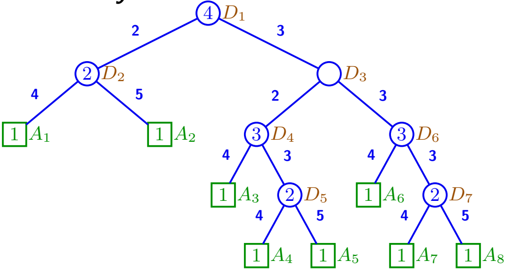


A Proof in Different Representations

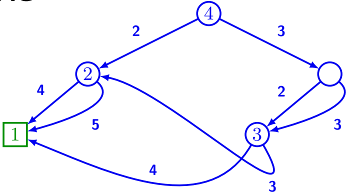
Connection Method Proof Structure



Full Binary Tree



DAG



Meredith's Notation

1. $CCCpqrCqr$
2. $CpCqp$ = D11
3. $CpCqCrp$ = D12

* 4. $CpCqCrCsCtCus$ = D2D33

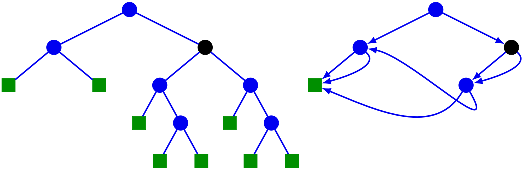
Meredith's 1963 Refinement of Łukasiewicz's 1948 Proof

1.	$CCCpqrCCrpCsp$	*Łukasiewicz*
2.	$CCCpqpCrp$ = DDD1D111n	
3.	$CCCpqrCqr$ = DDD1D1D121n	
4.	$CpCCpqCrq$ = D31	
5.	$CCCpqCrsCCCqtsCrs$ = DDD1D1D1D141n	
6.	$CCCpqCrsCCpsCrs$ = D51	
7.	$CCpCqrCCCpsrCqr$ = D64	
8.	$CCCCCpqrtCspCCrpCsp$ = D71	
9.	$CCpqCpq$ = D83	
10.	$CCCCrpCtpCCCpqrsCuCCCpqrs$ = D18	
11.	$CCCCpqrCsqCCCqtsCpq$ = DD10.10.n	
12.	$CCCCpqrCsqCCCqtpCsq$ = D5.11	
13.	$CCCCpqrsCCsqCpq$ = D12.6	
14.	$CCCpqrCCrpp$ = D12.9	
15.	$CpCCpqq$ = D3.14	
16.	$CCpqCCCprqq$ = D6.15	
* 17.	$CCpqCCqrCpr$ = DD13.D16.16.13	*Syll*
* 18.	$CCCpqpp$ = D14.9	*Peirce*
* 19.	$CpCqp$ = D33	*Simp*

Formal Basis – Condensed Detachment: Proof Structure as Term/Tree

D-term: Term with the binary function symbol D used to represent a proof structure as full binary tree

$\mathsf{D}(\mathsf{D}(1,1),\mathsf{D}(\mathsf{D}(1,\mathsf{D}(1,1)),\mathsf{D}(1,\mathsf{D}(1,1))))$



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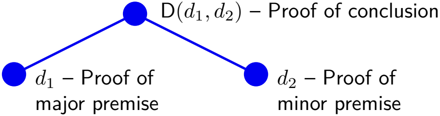


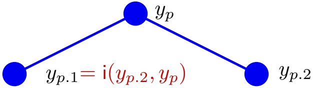
- Two useful size measures
  - ***Tree size***: number of inner nodes
  - ***Compacted size***: number of inner nodes of the minimal **DAG** = number of distinct compound subterms

Condensed Detachment: Substitutions, Proofs

- ***Axiom assignment*** α from primitive D-terms to formulas, e.g.

$$\{1 \mapsto \textit{Łukasiewicz}\}$$

- Global variable names y_p indexed with position p



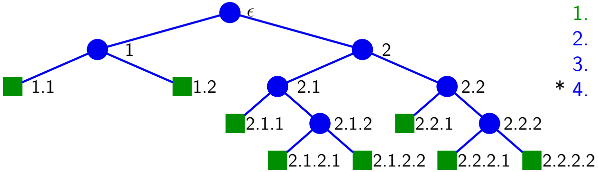
$$\textit{Detachment} \equiv \forall y_{p.2} \forall y_p (\mathsf{P}(y_p) \leftarrow \mathsf{P}(\mathsf{i}(y_{p.2}, y_p)) \land \mathsf{P}(y_{p.2}))$$

$$\mathsf{pairing}_\alpha(d, p) \overset{\text{def}}{=} \{y_p,\ \textit{copy for } p \textit{ of arg. of } \alpha(d|_p)\} \quad \text{if } p \in \mathcal{L}eaf\mathcal{P}os(d)$$
$$\mathsf{pairing}_\alpha(d, p) \overset{\text{def}}{=} \{y_{p.1},\ \mathsf{i}(y_{p.2}, y_p)\} \quad \text{if } p \in \mathcal{I}nner\mathcal{P}os(d)$$

- A ***proof*** is a D-term d together with an axiom assignment α s.th.

$$\{\mathsf{pairing}_\alpha(d, p) \mid p \in \mathcal{P}os(d)\} \text{ has an MGU}$$

Fine print: MGUs are assumed (i) idempotent (= domain and range are disjoint) (ii) variables in domain and range are in the unified terms [Eder 1985]



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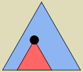
Condensed Detachment: Specific Formulas Associated with Positions

The ***in-place theorem (IPT)*** of d at p for α is

$$Ipt_\alpha(d,p) \stackrel{\text{def}}{=} \mathsf{P}(y_p\,\mathsf{mgu}(\{\mathsf{pairing}_\alpha(d,q) \mid q \in pos(d)\}))$$

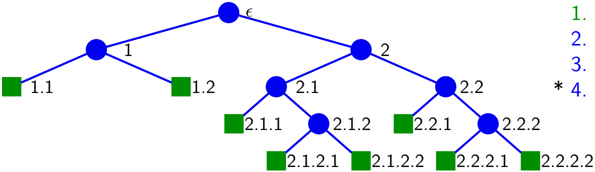


The ***most general theorem (MGT)*** of d for α is

$$Mgt_\alpha(d) \stackrel{\text{def}}{=} Ipt_\alpha(d,\epsilon)$$

$$Ipt_\alpha(d,p) \succeq Mgt_\alpha(d|_p)$$

$$\textit{Detachment} \land \bigwedge_{p \in \mathcal{L}eaf\mathcal{P}os(d)} \forall \mathsf{P}(\alpha(d|_p)) \;\models\; \forall Mgt_\alpha(d)$$



1. $CCCpqrCqr$
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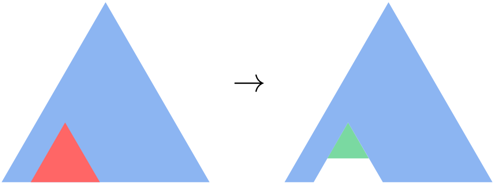
Reducing the Proof Size by Replacing Subproofs: Two Purposes



1. For a **given proof**:
   - Shorten it with respect to
     - tree size
     - compacted size
   - Determine irreducibility, that it can not be shortened in some specific way
2. In **proof search**:
   - Detect the redundancy of a proof fragment because a shorter one exists

Subproof Replacements that Reduce the Compacted Size

- Compacted size: size of minimal DAG = number of distinct subterms
- Technical basis: **comparison by set of distinct proper compound subterms**

$$d \geq_{\mathrm{c}} e \overset{\text{def}}{=} \{f \textit{ is compound} \mid d \rhd f\} \supseteq \{f \textit{ is compound} \mid e \rhd f\}.$$

$$\begin{array}{lcl} \mathsf{D}(\mathsf{D}(\mathsf{D}(1,1),1),1) & \geq_{\mathrm{c}} & \mathsf{D}(1,\mathsf{D}(1,1)) \\ \{\mathsf{D}(1,1), \mathsf{D}(\mathsf{D}(1,1),1)\} & \supseteq & \{\mathsf{D}(1,1)\} \end{array}$$

Theorem: Replacements that reduce the compacted size

If d' is d with **all** occurrences of e replaced by e', and $e \geq_{\mathrm{c}} e'$, then
$$\text{compacted-size}(d) \geq \text{compacted-size}(d').$$

- **Strict** version holds for $\text{sc-size}(d) \overset{\text{def}}{=} \sum_{d \trianglerighteq e} \text{compacted-size}(e)$
- Inspecting all $>_{\mathrm{c}}$-smaller proof structures is practically **feasible**:
  $|\{e \mid d >_{\mathrm{c}} e\}| \leq \text{compacted-size}(d+k)^2 + k$

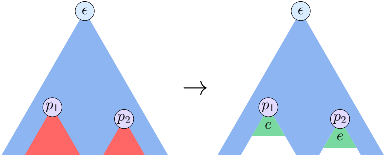
Subproof Replacement Based on IPT and MGT

$Ipt_\alpha(d,p) \stackrel{\text{def}}{=} \mathsf{P}(y_p \mathsf{mgu}(\{\mathsf{pairing}_\alpha(d,q) \mid q \in pos(d)\}))$; $\quad Mgt_\alpha(d) \stackrel{\text{def}}{=} Ipt_\alpha(d,\epsilon)$

Theorem: Subproof replacement based on the IPT

Let $p_1, \ldots, p_n$ be a set of positions of d. If e is such that for **all** $i \in \{1, \ldots, n\}$ it holds that $Ipt_\alpha(d, p_i) \geq Mgt_\alpha(e)$, then

$$Mgt_\alpha(d) \geq Mgt_\alpha(d[e]_{p_1}[e]_{p_2} \ldots [e]_{p_n}).$$



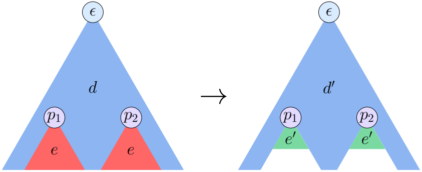
Theorem: Subproof replacement based on the MGT

If $Mgt_\alpha(d|_p) \geq Mgt_\alpha(e)$, then

$$Mgt_\alpha(d) \geq Mgt_\alpha(d[e]_p).$$

- Stronger precondition: $Ipt_\alpha(d,p) \geq Mgt_\alpha(d|_p)$
- Precondition independent of context structure; can be applied iteratively

Reductions and Regularities



- ***C-reduction***: Combination of the subproof replacement to reduce the compacted size with the semantics/substitution-related criteria based on the in-place theorem
- ***C-regular***: C-reduction is not applicable
- Compared to regularity in tableaux, C-regularity is special in two respects, that may be of general interest, also independently:
  1. Comparison of two nodes on a branch with **IPT for the upper node** and **MGT for the lower node**
  2. Based not on relating two nested subproofs but on comparison of **all occurrences** of a subproof with respect to all proofs that are $>_{\mathrm{c}}$**-smaller**

N-Simplification

- Proofs may involve applications of *Detachment* where the conclusion is **independent from the minor premise**
- Any axiom can then serve as a trivial minor premise
- Meredith expresses this with n as second argument of the D-term

1. $CCCpqrCCrpCsp$
2. $CCCpqpCrp$ = $\mathsf{DDD1D111}\mathrm{n}$
3. $CCCpqrCqr$ = $\mathsf{DDD1D1D121}\mathrm{n}$
4. $CpCCpqCrq$ = $\mathsf{D31}$
5. $CCCpqCrsCCCqtsCrs$ = $\mathsf{DDD1D1D1D141}\mathrm{n}$

The ***n-simplification*** of d is obtained by replacing subterms

$$\mathsf{D}(d_1, d_2) \text{ with } \mathsf{D}(d_1, \mathrm{n})$$

if both have the same MGT

Properties of Łukasiewicz's Proof as Refined by Meredith

		M	DT	DC	DH	DK_L	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT_U	IT_M	IH_U	IH_M
1.	1	1	0	0	0	0	•	–	17	554	6	6	3	4	•	•	0	0	4451	203	18	11
2.	D11		1	1	1	1	•	1=1	1	45	8	7	4	5	•	•	1	1	1640	220	17	12
3.	D12		2	2	2	1	•	1◁	1	45	11	8	4	6	•	•	2	2	1881	252	17	12
4.	D31		3	3	3	2	•	▷1	1	45	5	5	4	4	•	•	3	3	689	92	16	11
5.	D4n	2	4	4	4	3	•	▷n	1	45	4	4	3	3	•	•	4	4	688	91	15	10
6.	D15		5	5	5	3	•	1◁	1	45	6	5	3	4	•	•	5	5	1667	198	15	10
7.	D16		6	6	6	3	•	1◁	1	45	7	6	4	5	•	•	6	6	1802	208	16	11
8.	D17		7	7	7	3	•	1◁	1	45	9	7	4	6	•	•	7	7	2648	303	16	11
9.	D81		8	8	8	3	•	▷1	1	45	5	5	4	4	•	•	8	8	1032	119	15	10
10.	D9n	3	9	9	9	3	•	▷n	5	45	4	4	3	3	•	•	9	9	1031	118	14	9
11.	D10.1	4	10	10	10	4	•	▷1	2	37	4	4	3	3	•	•	10	10	448	60	13	9
12.	D1.11		11	11	11	4	•	1◁	1	23	7	7	5	5	•	•	11	11	498	73	14	10
13.	D1.12		12	12	12	4	•	1◁	1	23	12	8	5	6	•	•	12	12	1157	168	14	10
14.	D1.13		13	13	13	4	•	1◁	1	23	10	9	6	7	•	•	13	⟦12,13⟧	1050	159	15	11
15.	D1.14		14	14	14	4	•	1◁	1	23	15	10	6	8	•	•	14	⟦12,14⟧	1657	246	15	11
16.	D15.1		15	15	15	4	•	▷1	1	23	9	8	5	6	•	•	15	⟦12,15⟧	684	100	14	10
17.	D16.n	5	16	16	16	4	•	▷n	2	23	8	7	4	5	•	•	16	⟦12,16⟧	683	99	13	9
18.	D17.1	6	17	17	17	4	•	▷1	3	18	7	6	3	4	•	•	17	⟦12,17⟧	395	56	12	8
19.	D18.11	7	28	18	18	5	–	▷	1	14	7	6	4	4	•	•	14	⟦12,14⟧	209	61	11	9
20.	D19.1	8	29	19	19	6	–	▷1	2	14	9	8	5	5	•	•	15	⟦12,15⟧	132	38	10	8
21.	D1.20	10	30	20	20	6	–	1◁	2	10	12	9	5	6	•	•	16	⟦12,16⟧	158	47	10	8
22.	D21.21		61	21	21	6	–	=	1	5	10	9	5	6	•	•	⟦23,33⟧	⟦12,17⟧	53	16	9	7
23.	D22.n	11	62	22	22	6	–	▷n	1	5	9	8	4	5	•	•	⟦23,34⟧	⟦12,18⟧	52	15	8	6
24.	D17.23	12	79	23	23	6	–	◁	2	5	9	8	4	5	•	•	⟦23,51⟧	⟦12,23⟧	57	16	7	5
25.	D24.18	13	97	24	24	6	–	▷	2	2	7	6	4	4	•	•	⟦23,69⟧	⟦12,24⟧	27	17	6	5
26.	D20.10	9	39	20	20	7	–	▷	2	4	3	2	2	2	•	–	8	6	27	7	6	4
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3	5	5	3	3	•	•	⟦23,91⟧	⟦12,25⟧	24	7	6	4
28.	D10.27	15	129	26	25	7	–	◁	1	2	3	3	3	2	•	•	⟦23,101⟧	⟦12,26⟧	19	12	6	5
29.	D18.28	16	147	27	26	7	–	◁	2	2	5	5	4	3	•	•	⟦23,36⟧	⟦12,26⟧	19	12	6	5
30.	D29.29		295	28	27	7	–	=	1	1	10	7	5	4	•	•	⟦23,239⟧	⟦12,27⟧	13	13	5	5
31.	D25.30		393	30	28	7	–	$<_c$	1	1	7	7	5	4	•	•	⟦23,121⟧	⟦12,29⟧	13	13	5	5
32.	D31.25	17	491	31	29	7	–	▷	0	1	5	5	3	3	•	•	⟦23,191⟧	⟦12,30⟧	5	5	3	3
33.	D27.26	18	159	26	25	7	–	▷	0	1	3	3	3	2	•	•	15	11	3	3	3	3
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	•	•	7	6	2	2	2	2

Subproofs Exposed in Meredith's Presentation

		M	DT	DC	DH	DK$_L$	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT$_K$	IT$_M$	IH$_K$	IH$_M$
1.	1	1	0	0	0	0	•	–	17	554												
2.	D11		1	1	1	1	•	1=1	1	45												
3.	D12		2	2	2	1	•	1◁	1	45												
4.	D31		3	3	3	2	•	▷1	1	45												
5.	D4n	2	4	4	4	3	•	▷n	1	45												
6.	D15		5	5	5	3	•	1◁	1	45												
7.	D16		6	6	6	3	•	1◁	1	45												
8.	D17		7	7	7	3	•	1◁	1	45	9	7	4	6	•	•	7	7	2048	305	10	11
9.	D81		8	8	8	3	•	▷1	1	45	5	5	4	4	•	•	8	8	1032	119	15	10
10.	D9n	3	9	9	9	3	•	▷n	5	45												
11.	D10.1	4	10	10	10	4	•	▷1	2	37												
12.	D1.11		11	11	11	4	•	1◁	1	23												
13.	D1.12		12	12	12	4	•	1◁	1	23												
14.	D1.13		13	13	13	4	•	1◁	1	23												
15.	D1.14		14	14	14	4	•	1◁	1	23												
16.	D15.1		15	15	15	4	•	▷1	1	23												
17.	D16.n	5	16	16	16	4	•	▷n	2	23												
18.	D17.1	6	17	17	17	4	•	▷1	3	18												
19.	D18.11	7	28	18	18	5	–	▷	1	14												
20.	D19.1	8	29	19	19	6	–	▷1	2	14												
21.	D1.20	10	30	20	20	6	–	1◁	2	10												
22.	D21.21		61	21	21	6	–	=	1	5												
23.	D22.n	11	62	22	22	6	–	▷n	1	5												
24.	D17.23	12	79	23	23	6	–	◁	2	5												
25.	D24.18	13	97	24	24	6	–	▷	2	2												
26.	D20.10	9	39	20	20	7	–	▷	2	4												
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3												
28.	D10.27	15	129	26	25	7	–	◁	1	2												
29.	D18.28	16	147	27	26	7	–	◁	2	2												
30.	D29.29		295	28	27	7	–	=	1	1												
31.	D25.30		393	30	28	7	–	$<_c$	1	1												
32.	D31.25	17	491	31	29	7	–	▷	0	1												
33.	D27.26	18	159	26	25	7	–	▷	0	1												
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	•	•	7	6	2	2	2	2

M Line in Meredith's presentation
DD Number of incoming edges in the DAG
DR Number of occurrences

Meredith's Presentation

1. $CCCpqrCCrpCsp$
2. $CCCpqpCrp$ = DDD1D111n
3. $CCCpqrCqr$ = DDD1D1D121n
4. $CpCCpqCrq$ = D31
5. $CCCpqCrsCCCqtsCrs$ = DDD1D1D1D141n
6. $CCCpqCrsCCpsCrs$ = D51
7. $CCpCqrCCCpsrCqr$ = D64
8. $CCCCCpqrtCspCCrpCsp$ = D71
9. $CCpqCpq$ = D83
10. $CCCCrpCtpCCCpqrsCuCCCpqrs$ = D18
11. $CCCCpqrCsqCCCqtsCpq$ = DD10.10.n
12. $CCCCpqrCsqCCCqtpCsq$ = D5.11
13. $CCCCpqrsCCsqCpq$ = D12.6
14. $CCCpqrCCrpp$ = D12.9
15. $CpCCpqq$ = D3.14
16. $CCpqCCCprqq$ = D6.15

* 17. $CCpqCCqrCpr$ = DD13.D16.16.13
* 18. $CCCpqpp$ = D14.9
* 19. $CpCqp$ = D33

Dimensions of D-Terms

		M	DT	DC	DH	DK_L	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT_U	IT_M	IH_U	IH_M
1.	1	1	0	0	0	0	●	–	17	554	6	6	3	4			9	9	4451	203	18	11
2.	D11		1	1	1	1	●	1=1	1	45										220	17	12
3.	D12		2	2	2	1	●	$1\lhd$	1	45										252	17	12
4.	D31		3	3	3	2	●	$\rhd 1$	1	45										92	16	11
5.	D4n	2	4	4	4	3	●	$\rhd$n	1	45										91	15	10
6.	D15		5	5	5	3	●	$1\lhd$	1	45										198	15	10
7.	D16		6	6	6	3	●	$1\lhd$	1	45										208	16	11
8.	D17		7	7	7	3	●	$1\lhd$	1	45										303	16	11
9.	D81		8	8	8	3	●	$\rhd 1$	1	45										119	15	10
10.	D9n	3	9	9	9	3	●	$\rhd$n	5	45	4	4	3	3	●	●	9	9	1031	118	14	9
11.	D10.1	4	10	10	10	4	●	$\rhd 1$	2	37	4	4	3	3	●	●	10	10	448	60	13	9
12.	D1.11		11	11	11	4	●	$1\lhd$	1	23	7	7	5	5	●	●	11	11	498	73	14	10
13.	D1.12		12	12	12	4	●	$1\lhd$	1	23	12	8	5	6	●	●	12	12	1157	168	14	10
14.	D1.13		13	13	13	4	●	$1\lhd$	1	23	10	9	6	7	●	●	13	⟦12,13⟧	1050	159	15	11
15.	D1.14		14	14	14	4	●	$1\lhd$	1	23	15	10	6	8	●	●	14	⟦12,14⟧	1657	246	15	11
16.	D15.1		15	15	15	4	●	$\rhd 1$	1	23	9	8	5	6	●	●	15	⟦12,15⟧	684	100	14	10
17.	D16.n	5	16	16	16	4	●	$\rhd$n	2	23	8	7	4	5	●	●	16	⟦12,16⟧	683	99	13	9
18.	D17.1	6	17	17	17	4	●	$\rhd 1$	3	18	7	6	3	4	●	●	17	⟦12,17⟧	395	56	12	8
19.	D18.11	7	28	18	18	5	–	$\rhd$	1	14	7	6	4	4	●	●	14	⟦12,14⟧	209	61	11	9
20.	D19.1	8	29	19	19	6	–	$\rhd 1$	2	14	9	8	5	5	●	●	15	⟦12,15⟧	132	38	10	8
21.	D1.20	10	30	20	20	6	–	$1\lhd$	2	10	12	9	5	6	●	●	16	⟦12,16⟧	158	47	10	8
22.	D21.21		61	21	21	6	–	=	1	5	10	9	5	6	●	●	⟦23,33⟧	⟦12,17⟧	53	16	9	7
23.	D22.n	11	62	22	22	6	–	$\rhd$n	1	5	9	8	4	5	●	●	⟦23,34⟧	⟦12,18⟧	52	15	8	6
24.	D17.23	12	79	23	23	6	–	$\lhd$	2	5	9	8	4	5	●	●	⟦23,51⟧	⟦12,23⟧	57	16	7	5
25.	D24.18	13	97	24	24	6	–	$\rhd$	2	2	7	6	4	4	●	●	⟦23,69⟧	⟦12,24⟧	27	17	6	5
26.	D20.10	9	39	20	20	7	–	$\rhd$	2	4	3	2	2	2	●	–	8	6	27	7	6	4
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3	5	5	3	3	●	●	⟦23,91⟧	⟦12,25⟧	24	7	6	4
28.	D10.27	15	129	26	25	7	–	$\lhd$	1	2	3	3	3	2	●	●	⟦23,101⟧	⟦12,26⟧	19	12	6	5
29.	D18.28	16	147	27	26	7	–	$\lhd$	2	2	5	5	4	3	●	●	⟦23,36⟧	⟦12,26⟧	19	12	6	5
30.	D29.29		295	28	27	7	–	=	1	1	10	7	5	4	●	●	⟦23,239⟧	⟦12,27⟧	13	13	5	5
31.	D25.30		393	30	28	7	–	$<_c$	1	1	7	7	5	4	●	●	⟦23,121⟧	⟦12,29⟧	13	13	5	5
32.	D31.25	17	491	31	29	7	–	$\rhd$	0	1	5	5	3	3	●	●	⟦23,191⟧	⟦12,30⟧	5	5	3	3
33.	D27.26	18	159	26	25	7	–	$\rhd$	0	1	3	3	3	2	●	●	15	11	3	3	3	3
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	●	●	7	6	2	2	2	2

DT Tree size of D-term

DC Compacted size of D-term

DH Height of D-term

Dimensions of MGTs and IPTs

		M	DT	DC	DH	DK$_L$	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT$_U$	IT$_M$	IH$_U$	IH$_M$
1.	1	1	0	0	0	0	•	–	17	554	6	6	3	4	•	•	0	0	4451	203	18	11
2.	D11		1	1	1	1	•	1=1	1	45	8	7	4	5	•	•	1	1	1640	220	17	12
3.	D12		2	2	2	1	•	1◁	1	45	11	8	4	6	•	•	2	2	1881	252	17	12
4.	D31		3	3	3	2	•	▷1	1	45	5	5	4	4	•	•	3	3	689	92	16	11
										45	4	4	3	3	•	•	4	4	688	91	15	10
										45	6	5	3	4	•	•	5	5	1667	198	15	10
										45	7	6	4	5	•	•	6	6	1802	208	16	11
										45	9	7	4	6	•	•	7	7	2648	303	16	11
										45	5	5	4	4	•	•	8	8	1032	119	15	10
										45	4	4	3	3	•	•	9	9	1031	118	14	9
										37	4	4	3	3	•	•	10	10	448	60	13	9
										23	7	7	5	5	•	•	11	11	498	73	14	10
										23	12	8	5	6	•	•	12	12	1157	168	14	10
										23	10	9	6	7	•	•	13	⟦12,13⟧	1050	159	15	11
										23	15	10	6	8	•	•	14	⟦12,14⟧	1657	246	15	11
										23	9	8	5	6	•	•	15	⟦12,15⟧	684	100	14	10
										23	8	7	4	5	•	•	16	⟦12,16⟧	683	99	13	9
18.	D17.1	6	17	17	17	4	•	▷1	3	18	7	6	3	4	•	•	17	⟦12,17⟧	395	56	12	8
19.	D18.11	7	28	18	18	5	–	▷	1	14	7	6	4	4	•	•	14	⟦12,14⟧	209	61	11	9
20.	D19.1	8	29	19	19	6	–	▷1	2	14	9	8	5	5	•	•	15	⟦12,15⟧	132	38	10	8
21.	D1.20	10	30	20	20	6	–	1◁	2	10	12	9	5	6	•	•	16	⟦12,16⟧	158	47	10	8
22.	D21.21		61	21	21	6	–	=	1	5	10	9	5	6	•	•	⟦23,33⟧	⟦12,17⟧	53	16	9	7
23.	D22.n	11	62	22	22	6	–	▷n	1	5	9	8	4	5	•	•	⟦23,34⟧	⟦12,18⟧	52	15	8	6
24.	D17.23	12	79	23	23	6	–	◁	2	5	9	8	4	5	•	•	⟦23,51⟧	⟦12,23⟧	57	16	7	5
25.	D24.18	13	97	24	24	6	–	▷	2	2	7	6	4	4	•	•	⟦23,69⟧	⟦12,24⟧	27	17	6	5
26.	D20.10	9	39	20	20	7	–	▷	2	4	3	2	2	2	•	–	8	6	27	7	6	4
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3	5	5	3	3	•	•	⟦23,91⟧	⟦12,25⟧	24	7	6	4
28.	D10.27	15	129	26	25	7	–	◁	1	2	3	3	3	2	•	•	⟦23,101⟧	⟦12,26⟧	19	12	6	5
29.	D18.28	16	147	27	26	7	–	◁	2	2	5	5	4	3	•	•	⟦23,36⟧	⟦12,26⟧	19	12	6	5
30.	D29.29		295	28	27	7	–	=	1	1	10	7	5	4	•	•	⟦23,239⟧	⟦12,27⟧	13	13	5	5
31.	D25.30		393	30	28	7	–	$<_c$	1	1	7	7	5	4	•	•	⟦23,121⟧	⟦12,29⟧	13	13	5	5
32.	D31.25	17	491	31	29	7	–	▷	0	1	5	5	3	3	•	•	⟦23,191⟧	⟦12,30⟧	5	5	3	3
33.	D27.26	18	159	26	25	7	–	▷	0	1	3	3	3	2	•	•	15	11	3	3	3	3
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	•	•	7	6	2	2	2	2

TT	Tree size of MGT
IT$_U$	Maximum tree size of IPTs
IT$_M$	Median tree size of IPTs

Recall: $Ipt_\alpha(d,p) \geq Mgt_\alpha(d|_p)$

Prover9: `assign(max_depth,7)`

C-Regularity

RC Is C-regular

		M	DT	DC	DH	DK$_L$	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT$_U$	IT$_M$	IH$_U$	IH$_M$
1.	1	1	0	0	0	0	•	–	17	554	6	6	3	4	•	•	0	0	4451	203	18	11
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	•	1=1	1	45	8	7	4	5	•	•	1	1	1640	220	17	12
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	1	•	1◁	1	45	11	8	4	6	•	•	2	2	1881	252	17	12
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	2	•	▷1	1	45	5	5	4	4	•	•	3	3	689	92	16	11
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	3	•	▷n	1	45	4	4	3	3	•	•	4	4	688	91	15	10
6.	D15		5	5	5	3	•	1◁	1	45	6	5	3	4	•	•	5	5	1667	198	15	10
7.	D16		6	6	6	3	•	1◁	1	45	7	6	4	5	•	•	6	6	1802	208	16	11
8.	D17		7	7	7	3	•	1◁	1	45	9	7	4	6	•	•	7	7	2648	303	16	11
9.	D81		8	8	8	3	•	▷1	1	45	5	5	4	4	•	•	8	8	1032	119	15	10
10.	D9n	3	9	9	9	3	•	▷n	5	45	4	4	3	3	•	•	9	9	1031	118	14	9
11.	D10.1	4	10	10	10	4	•	▷1	2	37	4	4	3	3	•	•	10	10	448	60	13	9
12.	D1.11		11	11	11	4	•	1◁	1	23	7	7	5	5	•	•	11	11	498	73	14	10
13.	D1.12		12	12	12	4	•	1◁	1	23	12	8	5	6	•	•	12	12	1157	168	14	10
14.	D1.13		13	13	13	4	•	1◁	1	23	10	9	6	7	•	•	13	⟦12,13⟧	1050	159	15	11
15.	D1.14		14	14	14	4	•	1◁	1	23	15	10	6	8	•	•	14	⟦12,14⟧	1657	246	15	11
16.	D15.1		15	15	15	4	•	▷1	1	23	9	8	5	6	•	•	15	⟦12,15⟧	684	100	14	10
17.	D16.n	5	16	16	16	4	•	▷n	2	23	8	7	4	5	•	•	16	⟦12,16⟧	683	99	13	9
18.	D17.1	6	17	17	17	4	•	▷1	3	18	7	6	3	4	•	•	17	⟦12,17⟧	395	56	12	8
19.	D18.11	7	28	18	18	5	–	▷	1	14	7	6	4	4	•	•	14	⟦12,14⟧	209	61	11	9
20.	D19.1	8	29	19	19	6	–	▷1	2	14	9	8	5	5	•	•	15	⟦12,15⟧	132	38	10	8
21.	D1.20	10	30	20	20	6	–	1◁	2	10	12	9	5	6	•	•	16	⟦12,16⟧	158	47	10	8
22.	D21.21		61	21	21	6	–	=	1	5	10	9	5	6	•	•	⟦23,33⟧	⟦12,17⟧	53	16	9	7
23.	D22.n	11	62	22	22	6	–	▷n	1	5	9	8	4	5	•	•	⟦23,34⟧	⟦12,18⟧	52	15	8	6
24.	D17.23	12	79	23	23	6	–	◁	2	5	9	8	4	5	•	•	⟦23,51⟧	⟦12,23⟧	57	16	7	5
25.	D24.18	13	97	24	24	6	–	▷	2	2	7	6	4	4	•	•	⟦23,69⟧	⟦12,24⟧	27	17	6	5
26.	D20.10	9	39	20	20	7	–	▷	2	4	3	2	2	2	•	–	8	6	27	7	6	4
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3	5	5	3	3	•	•	⟦23,91⟧	⟦12,25⟧	24	7	6	4
28.	D10.27	15	129	26	25	7	–	◁	1	2	3	3	3	2	•	•	⟦23,101⟧	⟦12,26⟧	19	12	6	5
29.	D18.28	16	147	27	26	7	–	◁	2	2	5	5	4	3	•	•	⟦23,36⟧	⟦12,26⟧	19	12	6	5
30.	D29.29		295	28	27	7	–	=	1	1	10	7	5	4	•	•	⟦23,239⟧	⟦12,27⟧	13	13	5	5
31.	D25.30		393	30	28	7	–	$<_c$	1	1	7	7	5	4	•	•	⟦23,121⟧	⟦12,29⟧	13	13	5	5
32.	D31.25	17	491	31	29	7	–	▷	0	1	5	5	3	3	•	•	⟦23,191⟧	⟦12,30⟧	5	5	3	3
33.	D27.26	18	159	26	25	7	–	▷	0	1	3	3	3	2	•	•	15	11	3	3	3	3
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	•	•	7	6	2	2	2	2

The “Organic” Property [Leśniewski, Tarski, Łukasiewicz] of the MGT

		M	DT	DC	DH	DK_L	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT_U	IT_M	IH_U	IH_M
1.													3	4	•	•	0	0	4451	203	18	11
													4	5	•	•	1	1	1640	220	17	12
													4	6	•	•	2	2	1881	252	17	12
													4	4	∘	•	3	3	689	92	16	11
													3	3	•	•	4	4	688	91	15	10
													3	4	•	•	5	5	1667	198	15	10
													4	5	•	•	6	6	1802	208	16	11
													4	6	•	•	7	7	2648	303	16	11
													4	4	∘	•	8	8	1032	119	15	10
10.													3	3	•	•	9	9	1031	118	14	9
11.	D10.1	4	10	10	10	4	•	▷1	2	37	4	4	3	3	•	•	10	10	448	60	13	9
12.	D1.11		11	11	11	4	•	1◁	1	23	7	7	5	5	•	•	11	11	498	73	14	10
13.	D1.12		12	12	12	4	•	1◁	1	23	12	8	5	6	•	•	12	12	1157	168	14	10
14.	D1.13		13	13	13	4	•	1◁	1	23	10	9	6	7	•	•	13	⟦12,13⟧	1050	159	15	11
15.	D1.14		14	14	14	4	•	1◁	1	23	15	10	6	8	•	•	14	⟦12,14⟧	1657	246	15	11
16.	D15.1		15	15	15	4	•	▷1	1	23	9	8	5	6	∘	•	15	⟦12,15⟧	684	100	14	10
17.	D16.n	5	16	16	16	4	•	▷n	2	23	8	7	4	5	•	•	16	⟦12,16⟧	683	99	13	9
18.	D17.1	6	17	17	17	4	•	▷1	3	18	7	6	3	4	•	•	17	⟦12,17⟧	395	56	12	8
19.	D18.11	7	28	18	18	5	–	▷	1	14	7	6	4	4	•	•	14	⟦12,14⟧	209	61	11	9
20.	D19.1	8	29	19	19	6	–	▷1	2	14	9	8	5	5	•	•	15	⟦12,15⟧	132	38	10	8
21.	D1.20	10	30	20	20	6	–	1◁	2	10	12	9	5	6	•	•	16	⟦12,16⟧	158	47	10	8
22.	D21.21		61	21	21	6	–	=	1	5	10	9	5	6	∘	•	⟦23,33⟧	⟦12,17⟧	53	16	9	7
23.	D22.n	11	62	22	22	6	–	▷n	1	5	9	8	4	5	•	•	⟦23,34⟧	⟦12,18⟧	52	15	8	6
24.	D17.23	12	79	23	23	6	–	◁	2	5	9	8	4	5	•	•	⟦23,51⟧	⟦12,23⟧	57	16	7	5
25.	D24.18	13	97	24	24	6	–	▷	2	2	7	6	4	4	•	•	⟦23,69⟧	⟦12,24⟧	27	17	6	5
26.	D20.10	9	39	20	20	7	–	▷	2	4	3	2	2	2	•	–	8	6	27	7	6	4
27.	D24.26	14	119	25	24	7	–	$>_c$	2	3	5	5	3	3	•	•	⟦23,91⟧	⟦12,25⟧	24	7	6	4
28.	D10.27	15	129	26	25	7	–	◁	1	2	3	3	3	2	•	•	⟦23,101⟧	⟦12,26⟧	19	12	6	5
29.	D18.28	16	147	27	26	7	–	◁	2	2	5	5	4	3	•	•	⟦23,36⟧	⟦12,26⟧	19	12	6	5
30.	D29.29		295	28	27	7	–	=	1	1	10	7	5	4	•	•	⟦23,239⟧	⟦12,27⟧	13	13	5	5
31.	D25.30		393	30	28	7	–	$<_c$	1	1	7	7	5	4	•	•	⟦23,121⟧	⟦12,29⟧	13	13	5	5
32.	D31.25	17	491	31	29	7	–	▷	0	1	5	5	3	3	•	•	⟦23,191⟧	⟦12,30⟧	5	5	3	3
33.	D27.26	18	159	26	25	7	–	▷	0	1	3	3	3	2	•	•	15	11	3	3	3	3
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	•	•	7	6	2	2	2	2

TO organic (•), weakly organic (∘)

P(t) is ***organic*** if there is no strict subterm t' of t such that P(t') is a theorem

Experiments: Dimensions of Various Proofs of *Syll* from *Łukasiewicz*

Lemmas	#	*Time*	*Prover*	*Time*	**DC**	**DT**
			Łukasiewicz*		32	435
			Meredith		31	491
			E 2.5	30 s	length: 91	
			Vampire 5.4.1 casc	128/22 s	length: 144	
			KRHyper*	1,610 s	73	2,122
			Prover9	37 s	94	304,890
			*Prover9**	37 s	83	8,217
PrimeCore(17)	17		*Prover9**	30 s	44	763

*	N-simplification was applied with reducing effect
DC	Compacted size of D-term
DT	Tree size of D-term

Experiments: *PrimeCore* Lemmas – Background

		M	DT	DC	DH	DK_L	DP	DS
1.	1	1	0	0	0	0	•	–
2.	D11		1	1	1	1	•	1=1
3.	D12		2	2	2	1	•	$1\lhd$
4.	D31		3	3	3	2	•	$\rhd 1$
5.	D4n	2	4	4	4	3	•	$\rhd$n
6.	D15		5	5	5	3	•	$1\lhd$
7.	D16		6	6	6	3	•	$1\lhd$
8.	D17		7	7	7	3	•	$1\lhd$
9.	D81		8	8	8	3	•	$\rhd 1$
10.	D9n	3	9	9	9	3	•	$\rhd$n
11.	D10.1	4	10	10	10	4	•	$\rhd 1$
12.	D1.11		11	11	11	4	•	$1\lhd$
13.	D1.12		12	12	12	4	•	$1\lhd$
14.	D1.13		13	13	13	4	•	$1\lhd$
15.	D1.14		14	14	14	4	•	$1\lhd$
16.	D15.1		15	15	15	4	•	$\rhd 1$
17.	D16.n	5	16	16	16	4	•	$\rhd$n
18.	D17.1	6	17	17	17	4	•	$\rhd 1$
19.	D18.11	7	28	18	18	5	–	$\rhd$
20.	D19.1	8	29	19	19	6	–	$\rhd 1$
21.	D1.20	10	30	20	20	6	–	$1\lhd$
22.	D21.21		61	21	21	6	–	=
23.	D22.n	11	62	22	22	6	–	$\rhd$n
24.	D17.23	12	79	23	23	6	–	$\lhd$
25.	D24.18	13	97	24	24	6	–	$\rhd$
26.	D20.10	9	39	20	20	7	–	$\rhd$
27.	D24.26	14	119	25	24	7	–	$>_c$
28.	D10.27	15	129	26	25	7	–	$\lhd$
29.	D18.28	16	147	27	26	7	–	$\lhd$
30.	D29.29		295	28	27	7	–	=
31.	D25.30		393	30	28	7	–	$<_c$
32.	D31.25	17	491	31	29	7	–	$\rhd$
33.	D27.26	18	159	26	25	7	–	$\rhd$
34.	D10.10	19	19	10	10	4	–	=

DP Is prime

A D-term d is ***prime*** can be defined equivalently as

1. Every compound subterm of d occurs just once
2. tree-size(d) = compacted-size(d)
3. tree-size(d) = height(d)
4. d is in the language generated by
   $\langle prime \rangle \ ::= \ 1 \mid \mathrm{D}(\langle prime \rangle, 1) \mid \mathrm{D}(1, \langle prime \rangle)$

Among all the subproofs

- subproof 18 is the largest prime proof
- all prime proofs are subproofs of subproof 18
- hence we call subproof 18 ***"prime core"***

Adding the MGTs of the **prime core and all its subproofs as lemmas** leads with Prover9 to a smaller proof

A way to **find subproof 18 from the given axiom**:

- prime of size 17
- MGT not subsumed by that of smaller prime
- number of variables is 4, like goal theorem

Experiments: *ProofSubproof* Lemmas – Results

Lemmas	#	*Time*	*Prover*	*Time*	**DC**	**DT**
			Łukasiewicz*		32	435
			Meredith		31	491
			E 2.5	30 s	length: 91	
			Vampire 5.4.1 casc	128/22 s	length: 144	
			KRHyper*	1,610 s	73	2,122
			Prover9	37 s	94	304,890
			*Prover9**	37 s	83	8,217
PrimeCore(17)	17		*Prover9**	30 s	44	763
ProofSubproof(93)	291	78 s	*Prover9**	3 s	51	1,405
ProofSubproof(93)	291	78 s	*CMProver*	2 s	**!30**	394
ProofSubproof(100)	330	94 s	*CMProver*	4 s	**!30**	535

*	N-simplification was applied with reducing effect
DC	Compacted size of D-term
DT	Tree size of D-term
!	Minimal values known so far

Experiments: *ProofSubproof* Lemmas – Background

#		M	DT	DC	DH	DK_L	DP	DS	DD	DR	TT	TC	TH	TV	TO	RC	MT	MC	IT_U	IT_M	IH_U	IH_M
1.	1	1	0	0	0	0	●	–	17	554	6	6	3	4	●	●	0	0	4451	203	18	11
2.	D11		1	1	1	1	●	1=1	1	45	8	7	4	5	●	●	1	1	1640	220	17	12
3.	D12		2	2	2	1	●	1◁	1	45	11	8	4	6	●	●	2	2	1881	252	17	12
4.	D31		3	3	3	2	●	▷1	1	45	5	5	4	4	●	●	3	3	689	92	16	11
5.	D4n	2	4	4	4	3	●	▷n	1	45	4	4	3	3	●	●	4	4	688	91	15	10
6.	D15		5	5	5	3	●	1◁	1	45	6	5	3	4	●	●	5	5	1667	198	15	10
7.	D16		6	6	6	3	●	1◁	1	45	7	6	4	5	●	●	6	6	1802	208	16	11
8.	D17		7	7	7	3	●	1◁	1	45	9	7	4	6	●	●	7	7	2648	303	16	11
9.	D81		8	8	8	3	●	▷1	1	45	5	5	4	4	●	●	8	8	1032	119	15	10
10.	D9n	3	9	9	9	3	●	▷n	5	45	4	4	3	3	●	●	9	9	1031	118	14	9
11.	D10.1	4	10	10	10	4	●	▷1	2	37	4	4	3	3	●	●	10	10	448	60	13	9
12.	D1.11		11	11	11	4	●	1◁	1	23	7	7	5	5	●	●	11	11	498	73	14	10
13.	D1.12		12	12	12	4	●	1◁	1	23	12	8	5	6	●	●	12	12	1157	168	14	10
14.	D1.13		13	13	13	4	●	1◁														
15.	D1.14		14	14	14	4	●	1◁														
16.	D15.1		15	15	15	4	●	▷1														
17.	D16.n	5	16	16	16	4	●	▷n														
18.	D17.1	6	17	17	17	4	●	▷1														
19.	D18.11	7	28	18	18	5	–	▷														
20.	D19.1	8	29	19	19	6	–	▷1														
21.	D1.20	10	30	20	20	6	–	1◁														
22.	D21.21		61	21	21	6	–	=														
23.	D22.n	11	62	22	22	6	–	▷n														
24.	D17.23	12	79	23	23	6	–	◁														
25.	D24.18	13	97	24	24	6	–	▷														
26.	D20.10	9	39	20	20	7	–	▷														
27.	D24.26	14	119	25	24	7	–	$>_c$														
28.	D10.27	15	129	26	25	7	–	◁														
29.	D18.28	16	147	27	26	7	–	◁														
30.	D29.29		295	28	27	7	–	=														
31.	D25.30		393	30	28	7	–	$<_c$														
32.	D31.25	17	491	31	29	7	–	▷														
33.	D27.26	18	159	26	25	7	–	▷														
34.	D10.10	19	19	10	10	4	–	=	0	1	2	2	2	2	●	●	7	6	2	2	2	2

DS Relationship between the subproofs of major and minor premise

Relationships

=	identical
◁/▷	**sub-/superterm**
$<_c$/$>_c$	related by compaction ordering (if $\neq, \ntriangleleft, \ntriangleright$)

Indicators for special subproofs

1	axiom
n	arbitrary, i.e. n after n-simplication

Experiments: *ProofSubproof* Lemmas – Results

Lemmas	#	*Time*	*Prover*	*Time*	**DC**	**DT**
			Łukasiewicz*		32	435
			Meredith		31	491
			E 2.5	30 s	length: 91	
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ProofSubproof(93)	291	78 s	*CMProver*	2 s	**!30**	394
ProofSubproof(100)	330	94 s	*CMProver*	4 s	**!30**	535

*****	N-simplification was applied with reducing effect
DC	Compacted size of D-term
DT	Tree size of D-term
!	Minimal values known so far

Experiments: Rewriting Subproofs to Reduce the Tree Size

Lemmas	#	*Time*	*Prover*	*Time*	**DC**	**DT**
			Łukasiewicz*		32	435
			Meredith		31	491
			E 2.5	30 s	length: 91	
			Vampire 5.4.1 casc	128/22 s	length: 144	
			KRHyper*	1,610 s	73	2,122
			Prover9	37 s	94	304,890
			*Prover9**	37 s	83	8,217
PrimeCore(17)	17		*Prover9**	30 s	44	763
ProofSubproof(93)	291	78 s	*Prover9**	3 s	51	1,405
ProofSubproof(93)	291	78 s	*CMProver*	2 s	**!30**	394
ProofSubproof(100)	330	94 s	*CMProver*	4 s	**!30**	535
			Reduction of '30/394'		48	**!191**

*	N-simplification was applied with reducing effect
DC	Compacted size of D-term
DT	Tree size of D-term
!	Minimal values known so far

Conclusion: Next Steps

⇒ Explore further the ideas in the experiments

- Separation into "prime lemmas" whose proof can not be factorized and top-level structure that combines these
- Bottom-up lemma generation driven by proof structures

⇒ Proof structures as objects, terms

- Further developed meta-level theory of proof structures
- Incorporating into proof search, e.g. [Veroff 2001]

⇒ Stronger proof structure compressions than DAGs
e.g. by tree grammars with parameterized non-terminals that correspond to non-unit lemmas

⇒ Immediate generalization possibilities of considered proof tasks

- Other, pre-1936, known single axioms for the implicational fragment
- Single axioms and small axiom sets for further logics [Ulrich 2001, 2016]
- 200 Condensed Detachment problems in the TPTP
- Term representation of binary first-order resolution proofs

⇒ Soon to come: an implemented toolkit written in SWI-Prolog

Conclusion: Summary

- New formal reconstruction of Condensed Detachment
  - Specialization of the Connection Method
  - Full binary trees as simple structure of proofs and also of object terms
- New aspects concerning the replacements of subproofs
  - Shortening the compressed representation
  - Relating local with context-dependent formula instantiation
- Analysis of Lukasiewicz's proof and Meredith's refinement
  - Improved their size by applying techniques inspired by the analysis
  - Suggestion of a benchmark for proof conversions and representations

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