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Heinrich Behmann's Contributions to  
Second-Order Quantifier Elimination  
from the View of Computational Logic

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# Heinrich Behmann's Contributions to Second-Order Quantifier Elimination from the View of Computational Logic

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**Abstract.** For relational monadic formulas (the Löwenheim class) second-order quantifier elimination, which is closely related to computation of uniform interpolants, projection and forgetting – operations that currently receive much attention in knowledge processing – always succeeds. The decidability proof for this class by Heinrich Behmann from 1922 explicitly proceeds by elimination with equivalence preserving formula rewriting. Here we reconstruct the results from Behmann's publication in detail and discuss related issues that are relevant in the context of modern approaches to second-order quantifier elimination in computational logic. In addition, we give a summary as well as a commented register of further relevant material in unpublished manuscripts by Behmann and in his correspondence, especially with Wilhelm Ackermann.

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## Part I

# Introduction – Contributions of Behmann's Habilitation Thesis from the View of Computational Logic





## 1 Introduction to Part I

The Habilitation thesis of Heinrich Behmann (1891–1970), published in 1922 in *Mathematische Annalen* [Beh22a], belongs, along with works by Löwenheim [Löw15] and Skolem [Sko19; Sko20], to the standard references on the decision problem for relational monadic first-order formulas (the Löwenheim class),<sup>1</sup> and also for the extension of this class by second-order quantification upon predicates. Early such references of [Beh22a] include [HA28, p. 77], [HA38, p. 95], and [HB34, p. 200], where also the methods by Behmann are reproduced [HB34, p. 193ff and p. 200ff]. A detailed historic account is provided in Church’s *Introduction to Mathematical Logic* [Chu56, §49, in particular p. 293]. The book by Church also presents variants of methods from [Beh22a].

Behmann’s early work up to 1921 is presented in historic context by Mancosu [Man99]. The focus there is the dissertation from 1918, but various issues concerning [Beh22a], in particular its embedding into the context of the Hilbert school, are also documented. The historic analysis of the development of logics in the period 1917–23 by Zach [Zac99] describes Behmann’s contributions. In particular, it is observed there that Behmann’s talk on 10 May 1921 at the *Mathematische Gesellschaft* in Göttingen on the topic of his Habilitation thesis seems the first documented use of the term *Entscheidungsproblem* (*decision problem*) [Zac99, p. 363]. A transcript and English translation of this talk, along with a comprehensive introduction, has recently been published by Mancosu and Zach [MZ15]. The first published explicit statement of the decision problem seems to be in [Beh22a, p. 166] (see [Zac99] for an English translation of the relevant passages).

Behmann reduces the decision problem for relational monadic formulas to the second-order quantifier elimination problem, that is, the problem to compute for a given second-order formula an equivalent first-order formula. In computational logic, second-order quantifier elimination [GSS08], with variants called *uniform interpolation*, *forgetting* and *projection*, is today an area with a wide variety of applications and techniques. Some of today’s advanced methods for second-order quantifier elimination are explicitly based on the so-called *Ackermann’s Lemma*, due to Wilhelm Ackermann [Ack35a], and involve equivalence preserving rewriting of formulas as key technique [DŁS97]. Although Behmann actually uses such rewriting techniques and gave with [Beh22a] at that time Ackermann the impetus to investigate the elimination problem – as Ackermann courteously remarked in a letter to Behmann dated 29 Oct 1934 [Letter L7]<sup>2</sup> (see also Sect. 27) – it appears that [Beh22a] so far has been largely overlooked in the context of second-order quantifier elimination in computational logic, such as the monograph [GSS08], with the exception of historic references in [Cra08; Sch12] and a recent paper by the present author [Wer15].

<sup>1</sup> Relational monadic formulas are first-order formulas with only unary predicates and no functions other than constants.

<sup>2</sup> Letters and manuscripts in Behmann’s bequest are listed in Part V.

In this report we provide a detailed technical reconstruction of the methods and results from [Beh22a] (Part II) and discuss various related issues, of which many are still today of relevance in computational logic (Part III). We summarize follow-up works by Behmann himself in unpublished manuscripts and in the correspondence with Wilhelm Ackermann, which mainly concerns elimination in presence of predicates with arity larger than one (Part IV). This is supplemented by commented listings of publications by Behmann and documents in his bequest that are related to second-order quantifier elimination (Part V). The correspondence with Wilhelm Ackermann, as far as archived in Behmann’s bequest in the *Staatsbibliothek zu Berlin*, is registered there completely. Part VI concludes the report.

We do not address another major concern of Behmann that is related to computational logic: his approach to resolve paradoxes, based on the idea that these emerge from unjustified elimination of shorthands (*Kurzzeichen*), leading to a variant of lambda conversion and restricted quantifiers [Beh31; Beh59]. He discussed his approach, which is briefly mentioned by Curry and Feys in [CFC58, p. 4, 9, 260f], in correspondence with, among others, Ackermann, Bernays, Church, Gödel and Ramsey.

As already indicated, Behmann’s Habilitation thesis [Beh22a] has so far mainly been considered in the context of the history of the decision problem. However, from the point of view of computational logic it is relevant also in various further respects, not merely for historical reasons, but there are also technical aspects that are still of significance today, for example, the successful termination of second-order quantifier elimination methods on relational monadic formulas [Wer15], as well as methodical aspects, such as the roles of normal forms. The remaining sections of this part discuss these contributions.

## 2 Specification of the Decision Problem

As already mentioned, the first explicit statement of the decision problem seems to be in [Beh22a]. For a translation of the relevant passages and discussions see [Zac99; MZ15].

## 3 Solution of the Decision Problem for Relational Monadic First- and Second-Order Formulas with Equality

As indicated above, this result was first obtained by Löwenheim, whereas Skolem and Behmann provided further proofs. Like Behmann’s method, the techniques of Löwenheim and Skolem also apply if predicate quantification is considered [Chu56, p. 293]. As further noted in [Chu56, p. 293], Behmann’s method to handle equality is similar to that of Skolem in some important respects, but seems to have been found independently. Behmann himself describes this in a letter dated 27 December 1927 to Heinrich Scholz [BehNL, Kasten 3, I 63], brought to

attention in [Man99] with excerpts published in [MZ15] and below – see p. 10 and 91. Skolem’s proof is outlined from the perspective of elimination in [Cra08]. A methodical aspect of [Beh22a] seems worth mentioning: The decision problem is attacked there by investigating decidability explicitly for specific *syntactically characterized formula classes* (*Aussagenbereiche*).

## 4 Specification of the Problem of Second-Order Quantifier Elimination

As described in [Cra08], elimination problems play an important role in the works of Boole [Boo54] and Schröder [Sch05]. It seems, however, that the problem of second-order quantifier elimination has not been fully understood and explicitly stated accordingly before [Beh22a]. The second-order quantifier elimination problem is called there a *new “elimination problem”* (*neue[s] „Eliminationsproblem“*) and is explicated in the context of the instance that occurs first in that paper, the elimination of a unary predicate with respect to a formula of relational monadic first-order logic without equality.<sup>3</sup> This specification can be paraphrased as follows: Given is a formula  $\forall p F$  or  $\exists p F$ , where  $p$  is a unary predicate and  $F$  is of monadic first-order logic without equality. The objective is now to find a, or – as can be said more determined – the (first-order) formula that is equivalent to the given formula – with respect to the predicates with exception of  $p$ , the constants and the free variables in  $F$  – but does not contain  $p$  any more. Behmann also gives a second more semantic view on the elimination problem: The (first-order) relationship among the predicates (with exception of  $p$ ), constants and free variables in  $F$  should be determined that is a necessary and sufficient condition for  $F$  being true for arbitrary predicates  $p$  or for at least one predicate  $p$ , respectively.

Following Schröder [Sch05], Behmann calls the formula sought after *resultant* (*Resultante*). In the context of his elimination method, Behmann speaks in early manuscripts from 1921 of *separation* (*Aussonderung*) instead of *Elimination*. For instance, on p. 13 in [Manuscript M1], a method description is headed *Eliminationsverfahren*. (*Aussonderung?*). In [Manuscript M3], the manuscript for [Beh22a], on p. 40, the specification of the elimination problem quoted in footnote 3 uses „*Aussonderungsproblem*“ in place of „*Eliminationsproblem*“, on p. 45 the originally typed term *Aussonderungshauptform* is altered by a handwritten annotation to *Eliminationshauptform* (German for *main form for elimination*).

<sup>3</sup> [Beh22a, p. 196f]: Ich möchte dieses neue „Eliminationsproblem“ in der folgenden Weise bestimmter fassen: Gegeben ist eine Aussage

$$\varphi F_{\varphi f g a b} \text{ oder } \bar{\varphi} F_{\varphi f g a b},$$

wo der Operand eine Aussage unseres früheren Bereiches  $A$  ist und, abgesehen von  $\varphi^x$ , nur konstante Eigenschaften, natürlich in beliebiger endlicher Anzahl, enthält – da sie nämlich innerhalb des obigen Ausdrucks nicht durch Operatoren vertreten sind, haben wir sie eben, solange wir unser Augenmerk nur auf diesen richten, als

Behmann [Beh22a, p. 218ff] remarks that Schröder distinguishes between “elimination problem” and “summation problem”, which are in Behmann’s view actually identical. Schröder’s elimination problem is, in Behmann’s words, to find a condition for the satisfiability of  $F$  that is free from  $p$ .<sup>4</sup> Schröder’s “summation problem” is, according to Behmann, to find a formula that is equivalent to  $\exists p F$  and is free from  $p$ . As Behmann describes, Schröder observed the equivalence of both problems in a note inserted during printing of the third volume of his *Vorlesungen über die Algebra der Logik* [Sch95, p. 489–490], whereas the actual identity of both problems escaped him through his concern for analogy with numerical algebra. Behmann concludes with commenting that this is a strange evidence for the extent in which for Schröder content receded in favor of form.<sup>5</sup>

Although Behmann explicitly formulates the problem of eliminating second-order quantifiers and reduces the decision problem for relational monadic formulas to that elimination problem, he remains skeptical on whether the generalization to predicates with arbitrary arities and higher-order concepts can still be based on the elimination problem. His argument in [Beh22a, p. 226f] is summarized in Part IV, p. 57. In his letter to Heinrich Scholz dated 27 December 1927 [BehNL, Kasten 3, I 63], answering Scholz’s question about who has written before him and, in particular, who has written at first, about the decision

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konstant anzusehen –, außerdem möglicherweise noch konstante und veränderliche Individuen; diese letzten natürlich zugleich als Operatoren, daher oben nicht angedeutet. (Das Auftreten von Aussagen  $p, q, \dots$  als Grundbestandteile [im Original: Grundbestandteilen] hat im gegenwärtigen Zusammenhang kaum praktische Bedeutung.) *Es soll nun eine oder – wie wir bestimmter sagen dürfen – diejenige Aussage (erster Ordnung) gefunden werden, die der gegebenen äquivalent ist – natürlich für irgendwelche Werte von  $f, g, a, b$  –, den Begriff  $\varphi^x$  jedoch nicht mehr enthält.* Mit anderen Worten: *Es soll diejenige Beziehung (erster Ordnung) zwischen den Konstanten  $f, g, a, b$  ermittelt werden, deren Bestehen die notwendige und hinreichende Bedingung dafür ist, daß  $F_{\varphi f g a b}$  je nachdem für beliebige Begriffe  $\varphi^x$  oder für mindestens einen Begriff  $\varphi^x$  eine richtige Aussage darstellt.*

<sup>4</sup> Another interpretation of Schröder’s concept of the elimination problem is to find a consequence of  $F$  that has exactly those formulas as consequences which are consequences of  $F$  and do not contain  $p$ . See, for example, [Sch91, p. 200].

<sup>5</sup> In modern view, a divergence between syntactic and semantic conceptions of elimination actually arises: For example, the quantified Boolean formula  $\exists p q \wedge p$  is equivalent to the propositional formula  $q \wedge (p \vee \neg p)$ , where the Boolean quantifier has been “eliminated”. However, the formula still contains syntactically the formerly quantified atom  $p$ , although, from a semantic point of view, redundantly. The characterization of such redundancy is not always evident, for example, for modern variants of second-order quantifier elimination, where it is possible to “quantify upon” just a particular ground atom. On the other hand, for first-order logic, both views coincide in a sense: As noted in [Ott00, Introduction], the construction of interpolants according to Craig’s interpolation theorem can be applied to compute for a given first-order formula that is known to be *equivalent* to a formula expressed with a certain signature (predicates, functions and constants), an equivalent formula that is *syntactically* in that signature. The existence of an equivalent formula in a given signature can be expressed as validity.

problem (Postcard from Scholz to Behmann, dated 19 December 1927, [BehNL, Kasten 3, I 63]), he relates the decision problem to the elimination problem and remarks that the latter has been treated “first by the Americans, in particular Peirce, and later with particular love and persistence by Schröder, and eventually found a specialist in Löwenheim, who wrote several treatises about it in the *Math. Annalen*”. Behmann continues that the most important of these works by Löwenheim is *Über Möglichkeiten im Relativkalkül* [Löw15] and mentions that Löwenheim came there already to important partial results of his work [Beh22a], “however – in a presentation that is neither mathematically strict nor sufficiently comprehensible, such that I have construed these properly only after publication of my own paper.”<sup>6</sup>

## 5 Solution of the Second-Order Quantifier Elimination Problem for Relational Monadic Formulas with Equality

Behmann [Beh22a] presents an effective method for eliminating the second-order quantifiers in a given relational monadic formula with equality and with predicate quantification. (The earlier decidability proofs in [Löw15] and [Sko19] mentioned above are similarly based on second-order quantifier elimination in monadic logics.) Behmann’s method terminates after a finite number of steps with an equivalent relational monadic *first-order* formula. The result formula might be with equality also in cases where the given formula is without equality. For a given formula in which all predicates are quantified, the result formula just expresses constraints on the domain cardinality: The formula is either true for all domain cardinalities with exception of a finite number or false for all domain cardinalities with exception of a finite number. Obviously, valid and unsatisfiable formulas are special cases of such formulas.

The key technique of Behmann’s procedure is to propagate quantifiers inward, also for the price of expensive operations such as distribution of conjunction over disjunction. As suggested by Behmann, we call the resulting form *innex*.<sup>7</sup> This inward propagation is applied to quantifiers upon individual variables as well as to quantifiers upon predicates. A detailed presentation of Behmann’s method is the topic of Part II, further aspects of the method will be discussed in Part III.

## 6 Clarification of Schröder’s Early Results on Elimination

Most issues solved in [Beh22a] have been raised by Schröder in his *Vorlesungen über die Algebra der Logik* [Sch05]. Their solutions are developed by Behmann in

<sup>6</sup> An excerpt of the German letter is quoted in Sect. 32, p. 91. Further parts of that letter are summarized in [MZ15].

<sup>7</sup> *Die Endform meines Reduktionsverfahrens wird gelegentlich (sprachlich wenig glücklich) als "kontraprænex" bezeichnet; ich ziehe die Benennung "innex" vor.* Letter from Heinrich Behmann to Alonzo Church, 30 January 1959 [BehNL, Kasten 1, I 11].

a more modern representational framework, with a dedicated notation for logics, not obfuscated by the aim for correspondence to numeric algebra. The work by Behmann seems thus also useful as a guide to Schröder’s results, complementing the outline in [Cra08]. We already sketched Behmann’s discussion of Schröder’s notion of elimination in Sect. 4. The precise relationship of Behmann’s core result to Schröder’s earlier partial result, in particular, his „*crude resultant*” („*Resultante aus dem Rohen*“), as described Behmann will be shown in Sect. 15.

## 7 Methodology: Computation by Equivalence Preserving Rewriting

With the requirement to decide a statement after a finite number of steps, the *Entscheidungsproblem* inherently involves some notion of computation. Computation steps are expressed in [Beh22a] as equivalence preserving rewriting steps of logical formulas, justified by a collection of formula equivalences. A method starts with a given formula. At each step, a subformula occurrence is replaced with an equivalent formula, according to some computation rule (*Rechenregel*), that is, an equivalence from the collection, oriented either from left to right or from right to left. Computation terminates if the formula has reached a specific syntactic form. As Behmann states [Beh22a, p. 167], the particular collection of equivalences he gives is motivated not by finding a small set of orthogonal axioms, but by satisfying the needs of practical computation (*Bedürfnisse des praktischen Rechnens*).<sup>8</sup>

The foundation on equivalence, a semantic property, ensures that equivalence to the originally given formula is maintained as an invariant throughout the computation. Today, the representation by rewriting rules or transition systems that preserve semantic properties is a well established elegant way to represent computational methods such that they can be analyzed. Second-order quantification allows to represent also notions like equi-satisfiability (two formulas are either both satisfiable or unsatisfiable) as equivalence of formulas, making the preservation of equivalence a particularly useful invariant.

The modern so-called *direct methods* or *methods of the Ackermann approach*, for second-order quantifier elimination [GSS08], initiated by [Sza93; DLS97], typically founded on Ackermann’s Lemma [Ack35a], quite literally follow Behmann’s template of applying equivalence preserving formula rewritings that include various formula conversions and elimination steps.

## 8 Methodology: Normal Forms

The methods introduced in [Beh22a] essentially operate by converting given logical formulas to equivalent formulas in specific normal forms, that is, formulas with specific syntactic properties. Conjunctive and disjunctive normal forms are

<sup>8</sup> See [Zac99, p. 351f] for an English translation of the relevant section from [Beh22a].

used dually there, the inner normal forms for quantifiers upon instance variables as well as upon predicates are developed and counting (or cardinality) quantifiers are applied (preceded in [Löw15; Sko19] – see [Cra08]). Behmann’s method rewrites formulas to a certain intermediate form that allows predicate elimination according to a simple scheme.

The syntax for formulas used in [Beh22a] is based on disjunction, conjunction and negation, exposing the symmetry and duality inherent in these operations, which is, as criticized by Behmann, obscured in notations based on implication used by Frege and in the *Principia Mathematica*. For monadic formulas in a certain normal form, Behmann introduces a special notation (*Klassensymbolik*), where individual variables are suppressed.

In modern computational logic, normal forms play various roles. Systems typically operate on inputs in conjunctive normal form, obtained from preprocessors. Normal forms that allow to perform certain operations in an inexpensive way are investigated as target formats for knowledge compilation. The preservation of a certain normal form by calculi is applied to ensure that outputs are in a certain fragment of first-order logic or can be mapped to some other logic such as a modal or description logic (the method of [KS13] for second-order quantifier elimination in description logics can be considered as an example). Normal forms can provide representations of formulas that facilitate to understand their meaning, which is useful in the development of techniques as well as to present results to end users. This aspect of getting an overview on a solution in its totality has been a continuous concern for Behmann, for example in his comments to Ackermann’s resolution-based elimination technique, summarized below in Part IV, or in his later work on the solution problem (*Auflösungsproblem*) [Beh50; Beh51].

The paradigm of “model computation” or “answer set programming” in automated reasoning and logic programming can be considered as a variant of normal form computation: Such systems enumerate data structures that represent models. Their solutions can be regarded as a normalized representation of the input. A particular special case is enumerating all conjunctive clauses of a disjunctive normal form.

The so-called quantifier elimination approach in early model theory of the 1920s is another area where variants of normal form computation play an essential role, as will be discussed below in Sect. 13. The integration of such quantifier elimination methods for decidable theories into reasoning systems is currently an area of extensive research in automated deduction, motivated in particular by applications in software and hardware verification. Also the evaluation of relational database queries can be considered as elimination of first-order quantifiers [KKR95; Rev10].





## Part II

# Behmann's Decision and Elimination Method for Relational Monadic Formulas



## 9 Introduction to Part II

This part focuses on the main technical material in [Beh22a] from the point of view of second-order quantifier elimination as considered in computational logic. Modern notation is used throughout, on occasion a concordance to Behmann’s original labeling and various notations, which have merits on their own, are given. We aim here at a general formalization, where Behmann sometimes introduces techniques only with exemplary cases that provide intuition and indicate the general case. Also the structuring of the presentation deviates from the original paper, aiming at the modern reader.

The rest of this part is structured as follows: In Sect. 10 notation and terminology are introduced and general remarks on the presentation are given. Section 11 provides an overview on Behmann’s results, proceeding in a “top-down” fashion where the more involved methods and proofs are only sketched. With a collection of equivalences and entailments for use in formula rewriting and considerations on deciding and normalizing propositional logic, Sect. 12 paves the way for the more thorough presentation of Behmann’s techniques in the subsequent sections. First, the general case of monadic formulas *with equality* is considered. Sect. 13 describes a normalization method for such formulas. The second-order quantifier elimination method, which applies to the normalized formulas, is then shown in detail in Sect. 14. A simplified variant of the general method is then considered in Sect. 15. It applies just to the case without equality, but facilitates discussion of other issues, in particular the correspondence to earlier works by Schröder, as shown by Behmann.

## 10 Notational Conventions and Preliminary Remarks

### 10.1 Syntax

We briefly write *predicate*, *function* and *constant* for *predicate symbol*, *function symbol* and *constant symbol*, respectively. For second-order logic with equality, that is first-order logic with equality and extended by quantification upon predicates and functions, we use the following syntactic notation: An *atomic formula*, or briefly *atom*, is either of the form  $pt_1 \dots t_n$ , where  $n \geq 0$  and where  $p$  is a predicate of arity  $n$ , or of the form  $t_1 = t_2$ . In both cases, each subscripted  $t$  is a *term*, that is an *individual variable* or of the form  $ft_1 \dots t_n$ , where  $n \geq 0$ , the  $t_i$  are terms and  $f$  is a *function* of arity  $n$ .<sup>9</sup> A nullary function is also called *constant*. An atom of the form  $t_1 = t_2$  is called *equality atom*. Formulas and classes of formulas which may contain/may not contain equality atoms are called briefly *with equality* or *without equality*, respectively. If we speak of *first-order* formulas, unless explicitly indicated otherwise, we assume formulas with equality.

A *formula* is constructed from atoms, the constant operators  $\top$  (*true*),  $\perp$  (*false*), and a finite number of applications of the unary connectives  $\neg$  (*negation*), the binary connectives  $\wedge$  (*conjunction*) and  $\vee$  (*disjunction*), universal

<sup>9</sup> This parenthesis-free notation for terms and atoms is used in the modern textbook [EFT07].

quantification ( $\forall$ ) and existential quantification ( $\exists$ ). Negated equality  $\neq$ , further binary operators  $\rightarrow, \leftarrow, \leftrightarrow$ , as well as  $n$ -ary versions of  $\wedge$  and  $\vee$  can be understood as meta-level notation. The cases  $n = 0$ , which are  $\top$  and  $\perp$ , respectively, are included. The scope of  $n$ -ary operators in prefix notation is the immediate subformula to the right.

A subformula occurrence has in a given formula positive (negative) *polarity* if it is in the scope of an even (odd) number of negations. A *literal* is an atom or a negated atom. For a formula  $F$  of the form  $\neg G$ , the *complement*  $\bar{F}$  of  $F$  is  $G$ , if  $F$  has a form different from  $\neg G$ , then the complement is  $\neg F$ .

The scope of quantifiers is understood as extending as far to the right as possible. A quantifier occurrence may be upon an *individual variable* (“first-order”) or upon a *predicate* or *function* (“second-order”). We call the former also an *individual quantifier* and the latter *predicate quantifier*. An occurrence of an individual variable, predicate or function that is not bound by a quantifier occurrence in a formula is called *free* in that formula. As common in discussions of first-order logic, we distinguish between constants and free occurrences of individual variables. However, we do not make an analogous distinction for predicates and functions, since it would not be of relevance in the considered contexts. Thus, an occurrence of a predicate or function in a formula is just free or bound by a quantifier. In a first-order formula, all predicate and function occurrences are free.

## 10.2 Boolean Combination of Basic Formulas

Following patterns suggested by early model theory (see e.g. [CK90, Sect. 1.5]), in this presentation, several normal forms are characterized as *Boolean combination of basic formulas*, that is, as the formulas that are obtained from certain basic formulas, the constant operators  $\top$ ,  $\perp$  and repeated application of the operators  $\neg$ ,  $\wedge$  and  $\vee$ .<sup>10</sup>

## 10.3 Considered Formula Classes

We use the following symbols for particularly considered formula classes:  $\text{MON}$  is the class of relational monadic formulas (also called Löwenheim class), that is, the class of first-order formulas with nullary and unary predicates, with constants but no other functions, and without equality.  $\text{MON}_=$  is  $\text{MON}$  with equality.  $\text{QMON}$  and  $\text{QMON}_=$  are  $\text{MON}$  and  $\text{MON}_=$ , resp., extended by second-order quantification upon predicates.

All of these classes are decidable.  $\text{QMON}_=$  admits second-order quantifier elimination, that is, there is an effective method to compute for a given  $\text{QMON}_=$  formula  $F$  an equivalent  $\text{MON}_=$  formula  $F'$  in which all predicates are unquantified predicates in  $F$ , as well as all constants and free variables are also in  $F$ .

<sup>10</sup> This choice of operators has been made for convenience. Of course, technically it would be sufficient to just permit e.g.  $\neg$  and  $\wedge$ , and express  $\top$  and  $\perp$  as disjunction and conjunction, respectively, of an arbitrarily picked basic formula and its negation.

In this sense  $\text{MON}_=$  is closed under second-order quantifier elimination, which does not hold for  $\text{MON}$ , since elimination applied to a  $\text{QMON}$  formula might introduce equality.

A *quantified Boolean formula* is a formula where all predicates are nullary (called then also *Boolean variables* in the literature) and quantification is allowed just upon predicates.

#### 10.4 Remarks on the Presentation of Behmann’s Results

*Theorems that Assert the Existence of Effective Methods.* We formulate results often as theorem statements that assert the *existence of an effective method* to compute for a given formula an equivalent formula with certain properties. The proof is then typically a description of such a method. An alternative would be to just state the *existence of an equivalent formula with certain properties* as theorem. This course has not been followed here, since in contexts where the existence of a method is relevant, reference to the theorem alone (playing the role of a “module interface”) would not be sufficient, but the underlying proof (the “module implementation”) would have to be referenced.

*Free Individual Variables and Constants.* Many of the results of [Beh22a] apply to formulas – which may contain free variables – such that free variables and constants are handled in exactly the same way. While these results are presented here explicitly as properties of formulas, they are presented in [Beh22a] as properties of *sentences* (*Aussagen*), that is, formulas without free individual variables, considering free individual variables just as constants (see [Beh22a, footnote 25, p. 196]<sup>11</sup>).

*Consideration of Duality.* Methods based on equivalence preserving transformations of classical logic formulas typically come in two dual variants, where the roles of conjunction and disjunction as well as the roles of existential and universal quantification are switched (as made more precise with Prop. 7 in Sect. 12.1). In [Beh22a], such methods are in general explicitly developed for one of the variants and the dual variant is then indicated. In the presentation here, the discussion of dual variants is completely neglected, with exception of a few specific cases. Actually, from a technical point of view, the dual variants can be completely disregarded, since for inputs where they would seem adequate, their behavior would be just simulated by the original variant with the only difference that instead of atomic formulas (or basic formulas of other forms) their complements are used. As an example, consider the equivalence  $\exists x px \vee qx \equiv (\exists x px) \vee (\exists x qx)$  and its dual  $\forall x px \wedge qx \equiv (\forall x px) \wedge (\forall x qx)$ . The dual can be derived from the first variant with negated atoms in the following steps, which additionally only involve inward propagation of negation and expansion/contraction

<sup>11</sup> Behmann’s footnote translates as: I thus call the basic components of an expression “variable” („veränderlich“) or “constant” („konstant“), depending on whether it is represented within the expression by quantifiers (*Operatoren*) or not. As far as I can see, this is exactly the sense in which this distinction is actually made in mathematics.

of universal quantifiers:  $\forall x px \wedge qx \equiv \neg \exists x \neg (px \wedge qx) \equiv \neg \exists x \neg px \vee \neg qx \equiv \neg((\exists x \neg px) \vee (\exists x \neg qx)) \equiv (\neg \exists x \neg px) \wedge (\neg \exists x \neg qx) \equiv (\forall x px) \wedge (\forall x qx)$ .

## 11 Overview on Behmann’s Results and Methods

The final result of [Beh22a] can be stated as a theorem about  $\text{MON}_=$  formulas of a certain syntactic form. It allows to derive various results on second-order quantifier elimination and decidability of monadic formulas, including the decidability of  $\text{MON}$ . In this overview section we start with presenting this theorem and sketch in a “top-down” manner the techniques used in [Beh22a] to prove it. We then present the derived results, proven as corollaries of the theorem. More in-depth proofs of the theorem itself and discussions of the involved techniques will then be provided in a “bottom-up” manner in subsequent sections.

### 11.1 Elimination Method

The core result of [Beh22a] can be stated as follows:

**Theorem 1 (Predicate Elimination for  $\text{MON}_=$ ).** *There is an effective method to compute from a given predicate  $p$  and  $\text{MON}_=$  formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a  $\text{MON}_=$  formula,
2.  $F' \equiv \exists p F$ ,
3.  $p$  does not occur in  $F'$ ,
4. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

The proof given in [Beh22a] for Theorem 1 resides on the conversion of arbitrary  $\text{MON}_=$  formulas to a certain syntactic normal form that only allows restricted use of quantification. In particular, the scopes of quantifier occurrences are not permitted to overlap. To achieve this property, it is utilized that equality atoms with quantified variables can be represented implicitly by counting quantifiers  $\exists^{\geq n}$ , which express existence of at least  $n$  individuals. Formulas with counting quantifiers can be expanded into equivalent formulas of particular shapes with standard first-order quantifiers and equality atoms, as discussed in more detail below in Sect. 13.1. The standard quantifier  $\exists$  can be equivalently expressed as  $\exists^{\geq 1}$ . In the considered normal form, the argument formula of a counting quantifier  $\exists^{\geq n}$  must be a conjunction of literals of applications of a unary predicate to the quantified individual variable. The representability of  $\text{MON}_=$  formulas in this normal form can be stated as follows:

**Theorem 2 (Counting Quantifier Normal Form for  $\text{MON}_=$ ).** *There is an effective method to compute from a given  $\text{MON}_=$  formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a Boolean combination of basic formulas of the form:
  - (a)  $p$ , where  $p$  is a nullary predicate,

- (b)  $pt$ , where  $p$  is a unary predicate and  $t$  is a constant or an individual variable,
  - (c)  $t = s$ , where each of  $t, s$  is a constant or an individual variable,
  - (d)  $\exists^{\geq n}x \bigwedge_{1 \leq i \leq m} L_i[x]$ , where  $n \geq 1$ ,  $m \geq 0$  and the  $L_i[x]$  are pairwise different and pairwise non-complementary positive or negative literals with a unary predicate applied to the individual variable  $x$ ,
2.  $F' \equiv F$ ,
  3. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

If the given formula  $F$  in Theorem 2 is without equality, the allowed basic formulas can be strengthened by excluding the case  $t = s$  (c) and restricting the case (d) to  $n = 1$ , such that the counting quantifier can be considered as standard quantifier. The method of [Beh22a] to compute the normal form according to Theorem 2 proceeds by applying equivalence preserving formula rewritings to move quantifiers inward such that their scopes do not overlap. All predicate occurrences in the scope of a quantifier then have exactly the quantified variable as argument.

To achieve this, aside from inexpensive transformations such as narrowing quantifier scopes to subformulas where the quantified variables actually occur, distribution of existential (universal, resp.) quantifiers over disjunction (conjunction, resp.), propagating negation inward, and rearranging binary connectives according to associativity and commutativity, also *expensive transformations*, in particular distribution of conjunction over disjunction and vice versa, as familiar from conversion to disjunctive and conjunctive normal form, are required. Consider the following example, where initially  $\exists y$  is in the scope of  $\exists x$ :

$$\begin{aligned}
& \exists x px \wedge (qx \vee \exists y ry) \\
\equiv & \exists x (px \wedge qx) \vee (px \wedge \exists y ry) \\
\equiv & (\exists x px \wedge qx) \vee (\exists x px \wedge \exists y ry) \\
\equiv & (\exists x px \wedge qx) \vee ((\exists x px) \wedge (\exists y ry)).
\end{aligned} \tag{1}$$

Equality literals require special handling, as described in the full exposition in Sect. 13.2 below.

We now sketch a method as asserted by Theorem 1 for the case where the input formula is without equality. Given is the formula  $\exists p F$ , where  $F$  is a MON formula and  $p$  is a predicate. We only consider unary  $p$  here, nullary  $p$  could be handled analogously as a particularly simple special case. The input formula is first rewritten to an equivalent formula in which all occurrences of  $p$  are in subformulas of a specific syntactic form, called *Eliminationshauptform* (main form for elimination) in [Beh22a]:

$$\begin{aligned}
& \exists p \bigwedge_{1 \leq i \leq a} (\forall x A_i[x] \vee px) \quad \wedge \\
& \bigwedge_{1 \leq i \leq b} (\forall x B_i[x] \vee \neg px) \quad \wedge \\
& \bigwedge_{1 \leq i \leq c} (\exists x C_i[x] \wedge px) \quad \wedge \\
& \bigwedge_{1 \leq i \leq d} (\exists x D_i[x] \wedge \neg px),
\end{aligned} \tag{2}$$

where  $a, b, c, d$  are natural numbers  $\geq 0$  and the  $A_i[x], B_i[x], C_i[x], D_i[x]$  are first-order formulas in which  $p$  does not occur. This can be achieved with

the following steps: Normalize with the method of Theorem 2 and convert to disjunctive normal form, generalized such that the role of atoms is played by the basic formulas. Rewrite occurrences of  $pt$ , where  $t$  is a constant or a variable that is free in  $F$ , first with the equivalence  $\neg pt \equiv \forall x x \neq t \vee \neg px$  and then with  $pt \equiv \forall x x \neq t \vee px$ . This step introduces equality, which thus may also be present in the result. If  $F$  is without equality, the counting quantifiers are decorated with 1, directly corresponding to standard quantifiers. They can thus be rewritten with the equivalence  $\neg \exists^{\geq 1} x \bigwedge_{1 \leq i \leq m} L_i[x] \equiv \forall x \bigvee_{1 \leq i \leq m} \overline{L_i[x]}$ , and then with  $\exists^{\geq 1} x \bigwedge_{1 \leq i \leq m} L_i[x] \equiv \exists x \bigwedge_{1 \leq i \leq m} L_i[x]$ . After the predicate quantifier upon  $p$  is then moved inward with the techniques outlined for individual variable quantifiers in the context of Theorem 2, all occurrences of  $p$  are in subformulas that match the *Eliminationshauptform*.

The *Eliminationshauptform* allows to move the existential individual quantifiers and the constituents  $C_i$  and  $D_i$  to the front of the predicate quantifier, while the occurrences of  $p$  with existentially quantified arguments can be rewritten to universally quantified occurrences that match the forms  $\forall x A_i[x] \vee px$  or  $\forall x B_i[x] \vee \neg px$ , respectively, by applying the equivalences  $pt \equiv \forall x x \neq t \vee px$  and  $\neg pt \equiv \forall x x \neq t \vee \neg px$ . (Notice that in this step again equality may be introduced also in cases where the original formula  $F$  is without equality.) In this way, a formula in *Eliminationshauptform* can be further converted such that  $p$  only occurs in a subformula which is in a restricted form of the *Eliminationshauptform* allowing only the two universally quantified constituents, that is,  $c = d = 0$ . When restricted in this way, the *Eliminationshauptform* matches the left side of the following *Basic Elimination Lemma*, which gives a first-order equivalent for second-order formulas matching its left side:

**Lemma 3 (Basic Elimination Lemma).** *Let  $p$  be a unary predicate and let  $F, G$  be first-order formulas with equality in which  $p$  does not occur. It then holds that*

$$\exists p (\forall x F \vee px) \wedge (\forall x G \vee \neg px) \equiv \forall x F \vee G.$$

Note that  $F$  and  $G$  in that proposition may contain free variables, in particular free occurrences of  $x$ , which are then bound by the surrounding universal quantifiers on the left as well as on the right side of the proposition. Rewriting all subformulas headed with  $\exists p$  according to Lemma 3 then completes the method asserted by Theorem 1.

## 11.2 Applications to Predicate Elimination and Decidability

We now turn to applications of Theorem 1. Repeatedly running the method ensured by that theorem allows to eliminate *all* predicate quantifiers in a  $\text{MON}_=$  formula. The result is first-order, but might be with equality even in cases where the input is without equality, since a single run of the method already might introduce equality. This transfer of Theorem 1 to formulas with several predicate quantifiers is made precise with the following corollary:

**Corollary 4 (Elimination in  $\text{QMON}_=$ ).** *There is an effective method to compute for a given  $\text{QMON}_=$  formula  $F$  of a formula  $F'$  such that*



1.  $F'$  is a  $\text{MON}_=$  formula,
2.  $F' \equiv F$ ,
3. Predicates that just occur bound by a second-order quantifier in  $F$  do not occur in  $F'$ ,
4. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* Return the result of applying the following equivalence preserving rewritings to  $F$ : First, exhaustively<sup>12</sup> rewrite subformula occurrences of the form  $\forall p G$  where  $p$  is a predicate with the equivalent  $\neg \exists p \neg G$ . Second, exhaustively rewrite subformula occurrences of the form  $\exists p G$  where  $p$  is a predicate and  $G$  is first-order with (i.e.  $\exists p G$  is an innermost second-order quantification) to the equivalent first-order formula obtained according to Theorem 1.  $\square$

Basic formulas of form (d) in Theorem 2 include the case where  $m = 0$ , that is,  $\exists^{\geq n} x \top$ . If  $F$  is without constants, without free individual variables and such that all predicate occurrences are quantified, then the result of applying the method according to Corollary 4 followed by normalization according to Theorem 2 must be a Boolean combination of basic formulas of just the form  $\exists^{\geq n} x \top$ , where  $n$  is a number  $\geq 1$ . A formula  $\exists^{\geq n} x \top$  is satisfied by exactly those interpretations whose domain has at least  $n$  distinct members. A formula  $\neg \exists^{\geq n} x \top$  by those whose domain has less than  $n$  members. As observed in [Beh22a], a Boolean combination of formulas of the form  $\exists^{\geq n} x \top$  with  $n \geq 1$  is either true for all domain cardinalities with exception of a finite number or false for all domain cardinalities with exception of a finite number. It is not hard to see that validity and satisfiability of Boolean combinations of formulas of the form  $\exists^{\geq n} x \top$  is decidable. We discuss this in more depth in Sect. 13.1 below.

The decidability problem for  $\text{QMON}_=$  can be reduced to the elimination problem. Hence, the decidability of  $\text{QMON}_=$ , and thus also of its subclasses  $\text{QMON}$ ,  $\text{MON}_=$  and  $\text{MON}$ , follows from Corollary 4, and thus indirectly from Theorem 1. The following corollary states this from two perspectives, validity and satisfiability.

**Corollary 5 (Decidability of  $\text{QMON}_=$ ).**

- (i) There is an effective method to decide whether a  $\text{QMON}_=$  formula is valid.
- (ii) There is an effective method to decide whether a  $\text{QMON}_=$  formula is satisfiable.

*Proof.* (5.i) Let  $F$  be the given  $\text{QMON}_=$  formula. Let  $p_1, \dots, p_n$  be all predicates with free occurrences in  $F$ , let  $x_1, \dots, x_m$  be the free individual variables in  $F$ , and let  $c_1, \dots, c_k$  be the constants in  $F$ . Let  $F'$  be the formula

$$\forall p_1 \dots \forall p_n \forall x_1 \dots \forall x_m \forall c_1 \dots \forall c_k F.$$

The formula  $F'$  is valid if and only if  $F$  is valid. When applied to  $F'$ , the method according to Corollary 4 followed by normalization according to Theorem 2 then

<sup>12</sup> By *rewriting exhaustively* we mean that in each step a subformula occurrence of the indicated form is replaced and the resulting overall formula is subjected again to rewriting, until it does no longer contain a subformula of the indicated form.

yields an equivalent formula  $F''$  which is a Boolean combination of formulas of the form  $\exists^{\geq n} x \top$ , where  $n \geq 1$ . Validity of such Boolean combinations can be decided.

(5.ii) Decidability of satisfiability follows trivially from the decidability of validity, since a QMON<sub>=</sub> formula is satisfiable if and only if its negation, which is also a QMON<sub>=</sub> formula, is not valid. However, it is also possible to express the involved intermediate steps directly in terms of satisfiability: Let  $F$  be the given formula. Let  $F'$  be the formula

$$\exists p_1 \dots \exists p_n \exists x_1 \dots \exists x_m \exists c_1 \dots \exists c_k F,$$

where the quantified predicates, variables and constants are as specified in the proof of Prop. 5.i Then  $F'$  is satisfiable if and only if  $F$  is satisfiable. As in the proof for the decidability of validity, when applied to  $F'$ , the method according to Corollary 4 followed by normalization according to Theorem 2 yields an equivalent formula  $F''$  which is a Boolean combination of formulas of the form  $\exists^{\geq n} x \top$ . Satisfiability of such Boolean combinations can be decided.  $\square$

## 12 Starting Points: Rewrite Rules and Deciding Propositional Logic

The methodical approach of [Beh22a] essentially consists in developing effective methods that operate by rewriting of formulas in an equivalence preserving way to certain normal forms. The rewriting is done according to a set of rules, that is, oriented equivalences. The involved normal forms are in particular Boolean combinations of certain basic formulas as well as disjunctive and conjunctive normal form, generalized such that the role of atoms is played by certain basic formulas. In this section, a collection of the relevant equivalences and entailments that are used as rules is presented. In addition, the relationship between clausal normal forms and decision methods is sketched for propositional logic and quantified Boolean formulas, along with some comments on history.

### 12.1 Equivalences and Entailments for Rewriting Formulas

Well-know equivalences and entailments between formulas are listed below as labeled propositions, such that they can be referenced in the sequel. Their choice is mainly motivated by their role in the methods of [Beh22a]. A concordance with the rule labels used in [Beh22a] is provided with Table 12.1 at the end of the section. The following Prop. 6 gathers equivalences between formulas. In [Beh22a], such equivalences are used as rules for *reversible* inferences (*Regeln für umkehrbare Schlüsse*): they can be applied oriented from left to right as well oriented from right to left to obtain a formula that is equivalent to a given formula, but has different syntactic properties.

**Proposition 6 (Equivalences Useful for Rewriting).** *We consider second-order logic with equality. For all formulas  $F, G, H$ , quantifiers  $Q \in \{\forall, \exists\}$ , individual variables or predicates  $v, w$  and binary connectives  $\otimes \in \{\wedge, \vee\}$  the following equivalences hold:*

*Interaction of negation with other operators*

- EQ 1  $\neg\neg F \equiv F$ .  
 EQ 2  $\neg(F \wedge G) \equiv \neg F \vee \neg G$ .  
 EQ 3  $\neg(F \vee G) \equiv \neg F \wedge \neg G$ .  
 EQ 4  $\neg\forall v F \equiv \exists v \neg F$ .  
 EQ 5  $\neg\exists v F \equiv \forall v \neg F$ .

*Associativity, commutativity and idempotence of conjunction and disjunction*

- EQ 6  $(F \otimes G) \otimes H \equiv F \otimes (G \otimes H)$ .  
 EQ 7  $F \otimes G \equiv G \otimes F$ .  
 EQ 8  $F \otimes F \equiv F$ .

*Interaction of truth values with other operators*

- EQ 9  $\neg\top \equiv \perp$ .  
 EQ 10  $\neg\perp \equiv \top$ .  
 EQ 11  $\top \wedge F \equiv F$ .  $F \wedge \top \equiv F$ .  
 EQ 12  $\perp \wedge F \equiv \perp$ .  $F \wedge \perp \equiv \perp$ .  
 EQ 13  $\top \vee F \equiv \top$ .  $F \vee \top \equiv \top$ .  
 EQ 14  $\perp \vee F \equiv F$ .  $F \vee \perp \equiv F$ .  
 EQ 15  $Qv \top \equiv \top$ .  
 EQ 16  $Qv \perp \equiv \perp$ .

*Cancellation of complementary formulas*

- EQ 17  $F \wedge \neg F \equiv \perp$ .  $\neg F \wedge F \equiv \perp$ .  
 EQ 18  $F \vee \neg F \equiv \top$ .  $\neg F \vee F \equiv \top$ .

*Distribution among conjunction and disjunction*

- EQ 19  $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ .  
 $(F \vee G) \wedge H \equiv (F \wedge H) \vee (G \wedge H)$ .  
 EQ 20  $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$ .  
 $(F \wedge G) \vee H \equiv (F \vee H) \wedge (G \vee H)$ .

*Quantifier shifting*

- EQ 21  $(\forall v F) \wedge (\forall v G) \equiv \forall v F \wedge G$ .  
 EQ 22  $(\exists v F) \vee (\exists v G) \equiv \exists v F \vee G$ .  
 EQ 23  $(Qv F) \otimes G \equiv Qv F \otimes G$ , if  $v$  does not occur free in  $G$ .  
 $F \otimes (Qv G) \equiv Qv F \otimes G$ , if  $v$  does not occur free in  $F$ .

*Vacuous quantifiers, quantifier switching and variable renaming*

- EQ 24  $Qv F \equiv F$ , if  $v$  does not occur free in  $F$ .  
 EQ 25  $QvQw F \equiv QwQv F$ .  
 EQ 26  $QvF[v] \equiv QwF[w]$ , if  $F[v]$  and  $F[w]$  are identical with the exception that  $F[w]$  is obtained from  $F[v]$  by replacing all free occurrences of  $v$  with  $w$ , and vice versa,  $F[v]$  from  $F[w]$  by replacing all free occurrences of  $w$  with  $v$ .

*Absorption of entailed conjuncts and entailing disjuncts*

EQ 27  $F \wedge G \equiv F$ , if  $F \models G$ .

EQ 28  $F \vee G \equiv G$ , if  $F \models G$ .

*Clausal simplifications: tautology reduction, subsumption and unit reduction*

Here we consider a matrix (conjunction or disjunction with arity  $\geq 0$ ) of clauses (disjunctions or conjunctions, respectively, with arities  $\geq 0$ ) of basic formulas or negated basic formulas. Corresponding to the setting in [Beh22a], basic formulas are not restricted to atoms but can be arbitrary formulas (see also footnote 15 on p. 28). The indicated operations preserve equivalence of the matrix.

EQ 29 A clause that contains a basic formula and its complement can be removed.

EQ 30 A clause whose members are all contained in another clause can be removed.

EQ 31 A member of a clause can be removed if its complement is the sole member of another clause in the matrix.

*Circumlocution of argument terms*<sup>13</sup>

Let  $x$  be an individual variable, let  $F[x]$  be a formula, let  $t$  be a term that does not contain neither  $x$  nor a variable that is bound in  $F[x]$  (and thus is itself also different from  $x$  and from any variable bound in  $F[x]$ ), and let  $F[t]$  be  $F[x]$  with all free occurrences of  $x$  replaced by  $t$ . It then holds that

EQ 32  $F[t] \equiv \forall x x \neq t \vee F[x]$ .

EQ 33  $F[t] \equiv \exists x x = t \wedge F[x]$ .

Proposition 7 below also provides a basis for reversible inferences, but does not state an equivalence of formulas. Instead, it states an equivalence of statements about formulas. Let  $\text{dual}(F)$  denote the *dual* of a first- or second-order formula  $F$  (whose only operators are  $\top$ ,  $\perp$ ,  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\exists$ , and  $\forall$ ), that is, the formula obtained from  $F$  by switching  $\top$  with  $\perp$ ,  $\wedge$  with  $\vee$ , and  $\forall$  with  $\exists$ . The dual of  $F$  is equivalent to the negation of  $F$  after negating each atom occurrence.

**Proposition 7 (Preservation of Equivalence under Duality).** *Let  $F$  and  $G$  be first- or second-order formulas. It then holds that*

$$F \equiv G \text{ if and only if } \text{dual}(F) \equiv \text{dual}(G).$$

Proposition 8 shows entailments between formulas that are considered as rules for inferences that are not reversible (*Regeln für nicht umkehrbare Schlüsse*). Rewriting with them oriented from left to right yields a formula that is weaker than or equivalent to the original formula. This can be useful, for example, since validity is preserved by weaker formulas, since establishing the entailment relationship can give rise to equivalence preserving rewritings where the entailment is a precondition, and in the context of methods like resolution that proceed by enriching a given formula with entailed formulas.

<sup>13</sup> In Behmann's later manuscripts, the term *Umschreibung*, which might be translated as *circumlocution*, appears for the right sides of these equivalences.

**Proposition 8 (Entailments Useful for Rewriting).** *We consider formulas of first- and second-order logic.*

*For formulas  $F, G, H$  and first-order variable  $x$  it holds that*

$$(EN\ 34) \quad (F \vee G) \wedge (H \vee \neg G) \models F \vee H.$$

$$(EN\ 35) \quad (\forall x F \vee G) \wedge (\forall x H \vee \neg G) \models \forall x F \vee H.$$

*Let  $F[{}^+G]$  ( $F[{}^-G]$ , resp.) be a formula with an occurrence of subformula  $G$  with positive (negative, resp.) polarity. Let  $F[{}^+H]$  ( $F[{}^-H]$ , resp.) denote  $F$  with the occurrence of  $G$  replaced by formula  $H$ . It then holds that*

$$(EN\ 36) \quad F[{}^+G] \wedge (\neg G \vee H) \models F[{}^+H].$$

$$(EN\ 37) \quad F[{}^-G] \wedge (G \vee \neg H) \models F[{}^-H].$$

*Let  $F[{}^+G]$ ,  $F[{}^-G]$ ,  $G$ ,  $H$  be as specified before, with the exception that the first-order variables  $x_1, \dots, x_n$  possibly occur free in  $G$ ,  $H$  and also elsewhere in  $F[{}^+G]$  or  $F[{}^-G]$ , respectively. It then holds that*

$$(EN\ 38) \quad F[{}^+G] \wedge (\forall x_1 \dots \forall x_n \neg G \vee H) \models F[{}^+H].$$

$$(EN\ 39) \quad F[{}^-G] \wedge (\forall x_1 \dots \forall x_n G \vee \neg H) \models F[{}^-H].$$

Table 12.1 provides a concordance of the equivalences and entailments stated as propositions in this section and the computation rules (*Rechenregeln*) of [Beh22a]. Rules I–IV\*, VI, VII and IX–XI are equivalences of formulas. Rule VIII is an equivalence of statements about formulas. Rules V– $\bar{V}^*$  are entailments. The starred rules concern quantifiers. The correspondence given here is not in all cases one-to-one: The specification in [Beh22a] is only informal, such that we have to interpret some details in a specific way or quietly apply generalizations that are straightforward from today's point of view. Also, in [Beh22a] conjunction and disjunction are understood directly as  $n$ -ary operators, such that in some cases the effect of the rules from [Beh22a] can only be achieved by repeated application of the listed corresponding equivalences. Equivalences EQ 9–EQ 14 which concern embedded truth-value operators are not labeled in [Beh22a], but listed on p. 182 and 188.

## 12.2 Deciding Propositional Logic

As outlined in [Beh22a], a propositional formula  $F$  whose atoms are  $p_1, \dots, p_n$  is valid if and only if the quantified Boolean formula

$$\forall p_1 \dots \forall p_n F$$

is true, and satisfiable if and only if

$$\exists p_1 \dots \exists p_n F$$

is true. The Boolean quantifiers immediately indicate the substitution method (*Einsetzungsverfahren*) to decide propositional validity and satisfiability: Let

<sup>14</sup> The description of  $V^*$  in the appendix of [Beh22a] might suggest also the entailment  $(\exists x F \vee G) \wedge (\exists x H \vee \neg G) \models \exists x F \vee H$ , which does not hold in general.

<i>Label</i>	<i>Correspondence</i>	<i>Explanation</i>
I	EQ 1	double negation
II	EQ 6	associativity of conjunction and disjunction
II*	EQ 23	modifying the scope of quantifiers
III	EQ 7	commutativity of conjunction and disjunction
III*	EQ 25	permuting quantifiers of the same type
IV	EQ 8	idempotence of conjunction and disjunction
IV*	EQ 21, EQ 22	merging quantifiers over conjunctions and disjunctions
V	EN 34	generalized propositional resolvent
V*	EN 35	generalized propositional resolvent, with quantification <sup>14</sup>
$\bar{V}$	EN 36, EN 37	substitution of subformula by implied/implying formula
$\bar{V}^*$	EN 38, EN 39	like $\bar{V}$ , with quantification over the implication
VI	EQ 2, EQ 3	De Morgan's laws
VI*	EQ 4, EQ 5	negated quantifications
VII	EQ 20, EQ 19	distribution among conjunction and disjunction
VIII	Prop. 7	preservation of equivalence under duality
IX	EQ 30, EQ 31	subsumption and unit reduction in clausal forms
X	EQ 27, EQ 28	absorption of entailed conjuncts and entailing disjuncts
XI	EQ 32, EQ 33	circumlocution of argument terms

**Table 1.** Concordance with the rule labels in [Beh22a].

$F[p \mapsto G]$  denote  $F$  under substitution of all occurrences of atom  $p$  by  $G$ . Then  $\forall p F$  is true if and only if *for all* truth value constants  $G \in \{\top, \perp\}$  it holds that  $F[p \mapsto G] \equiv \top$ , while  $\exists p F$  is true if and only if *there exists* a truth value constant  $G \in \{\top, \perp\}$  such that  $F[p \mapsto G] \equiv \top$ . For nested quantifiers, the corresponding combined substitutions have to be evaluated.

A second method to decide propositional validity, attributed in [Beh22a] to Bernays [Ber18], consists in producing a conjunctive normal form, which is valid if and only if each of its clauses is valid, that is, contains a literal and its complement. In [Beh22a] it is noticed that propositional satisfiability can be analogously decided by conversion to disjunctive normal form:<sup>15</sup> A disjunctive normal form is satisfiable if and only if at least one of its (conjunctive) clauses is

<sup>15</sup> Actually, the names *konjunktive Normalform* and *disjunktive Normalform* as well as the precise realization of their duality seem due to [Beh22a] – see [Chu56, p. 166]. In [Ber18], the work referenced by [Beh22a], Bernays just speaks of *Normalform*, in the sense of conjunctive normal form. Footnote on p. 13 in [Ber18] suggests that the duality with disjunctive normal form was not common knowledge at that time: *Es wäre ein Irrtum, nach Analogie [...] zu vermuten, dass ein Produkt-Ausdruck dann und nur dann eine beweisbare Formel ist, wenn mindestens eines der Glieder eine beweisbare Formel ist.* (Can be paraphrased as: It would be an error to conjecture in analogy that a disjunction is a valid formula if and only if at least one of its disjuncts is a valid formula.) See also [Chu56, Note 299]. In [MZ15] it is observed that Hilbert considered conjunctive and disjunctive normal forms as well as decidability of propositional logic already in 1905. As further noted in [MZ15], Behmann writes on 27 December 1927 to Scholz [BehNL, Kasten 3, I 63] that, as far as he remembers, he

satisfiable, that is, does not contain a literal and its complement. As indicated in [Beh22a], quantified Boolean formulas that contain universal as well as existential quantifiers then could be evaluated by successive use of both normal forms. A third method to decide quantified Boolean formulas by moving quantifiers inward is also shown in [Beh22a]. It is sketched below in Sect. 17.

### 13 Counting Quantifier Normal Form for $\text{MON}_=$ Formulas

The method of [Beh22a] for the elimination of second-order quantifiers in  $\text{QMON}_=$  formulas, stated here as Corollary 4, based on Theorem 1, involves rewriting  $\text{MON}_=$  formulas to equivalent formulas that are constructed from a restricted set of basic formulas. In the target format, the only quantifiers permitted are counting quantifiers, and only in occurrences where their scopes are not nested.

The development of methods to express a class of formulas by Boolean combinations of certain basic formulas, typically with respect to a given background theory, is the core of *elimination of quantifiers*, the prevailing program of model theory in the 1920s (see e.g. [CK90, Sect. 1.5], [Hod97, Sect. 2.7]). Although [Beh22a] does not explicitly reference works clearly associated with that program, we present here the method to solve the “problem of normal form” (*Problem der Normalform*) given in [Beh22a, § 20] as instance of such a quantifier elimination method – with respect to the empty theory – following the template given in [CK90, Sect. 1.5].

The presentation in this section proceeds “bottom-up”. First, counting quantifiers, which are important as constituents of the basic formulas, are discussed, then the construction of the normal form is shown.

#### 13.1 Counting Quantifiers

As already indicated, formulas with counting quantifiers belong to the basic formulas of the envisaged target format. A counting quantifier  $\exists^{\geq n}x$ , where  $n$  is a natural number  $\geq 1$ , expresses existence of at least  $n$  individuals  $x$ . It is well known that these quantifiers can be defined in terms of standard first-order quantifiers and equality literals. There are two obvious possibilities to do so, which are both used in [Beh22a], as we will see below in Sect. 14. The following proposition lists them as equivalences:

**Proposition 9 (First-Order Expansions of Existential Counting Quantifiers).** *Let  $F[x]$  be a first-order formula which possibly has free occurrences of*

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had learned about the solution of the decision problem for propositional logic using normal forms directly from Hilbert.

As explained in [Beh22a, p. 185, footnote 19], Behmann uses *normal form* also more generally for a conjunction of disjunctions of arbitrary formulas, not necessary literals, or a disjunction of conjunctions of arbitrary formulas, respectively.

variable  $x$  and let  $n$  be a natural number  $\geq 1$ . Let  $x_1, \dots, x_n$  be distinct variables that are fresh (that is, different from  $x$  and not occurring in  $F[x]$ ), and, for  $i \in \{1, \dots, n\}$ , let  $F[x_i]$  denote  $F[x]$  with the free occurrences of  $x$  replaced by  $x_i$ . It then holds that

- (i)  $\exists^{\geq n} x F[x] \equiv \exists x_1 \dots \exists x_n \bigwedge_{1 \leq i \leq n} F[x_i] \wedge \bigwedge_{i < j \leq n} x_i \neq x_j$ .
- (ii)  $\exists^{\geq n} x F[x] \equiv \forall x_1 \dots \forall x_{n-1} \exists x F[x] \wedge \bigwedge_{1 \leq i < n} x \neq x_i$ .

The statement  $\exists^{\geq n} x \top$  expresses that the domain has at least  $n$  members. Further properties of  $\exists^{\geq n}$  are gathered in the following proposition:

**Proposition 10 (Properties of Existential Counting Quantifiers).** *For all first-order formulas  $F$  and natural numbers  $n, m \geq 1$  it holds that*

- (i)  $\exists^{\geq n} x \perp \equiv \perp$ .
- (ii)  $\exists^{\geq 1} x F \equiv \exists x F$ .
- (iii)  $\exists^{\geq 1} x \top \equiv \top$ .
- (iv)  $\exists^{\geq n} x F \models \exists^{\geq m} x F$ , if  $m \leq n$ .

Let  $\forall^{< n} x$ , where  $n$  is a natural number  $\geq 1$ , be further counting quantifiers, defined as shorthand for  $\neg \exists^{\geq n} x \neg$ . They express “for all with the exception of less than  $n$  individuals  $x$  it holds that”. In analogy to Prop. 9, they can be expanded in two way as follows:

**Proposition 11 (First-Order Expansions of Universal Counting Quantifiers).** *Let  $F[x], n, x_1, \dots, x_n, F[x_i]$  be as specified in Prop. 9. It then holds that*

- (i)  $\forall^{< n} x F[x] \equiv \forall x_1 \dots \forall x_n \bigvee_{1 \leq i \leq n} F[x_i] \vee \bigvee_{i < j \leq n} x_i = x_j$ .
- (ii)  $\forall^{< n} x F[x] \equiv \exists x_1 \dots \exists x_{n-1} \forall x F[x] \vee \bigvee_{1 \leq i < n} x = x_i$ .

The statement  $\forall^{< n} x \perp$  expresses that the domain has less than  $n$  members. Further properties of  $\forall^{< n}$  are gathered in the following proposition, analogously to Prop. 10:

**Proposition 12 (Properties of Universal Counting Quantifiers).** *For all first-order formulas  $F$  and natural numbers  $n, m \geq 1$  it holds that*

- (i)  $\forall^{< n} x \top \equiv \top$ .
- (ii)  $\forall^{< 1} x F \equiv \forall x F$
- (iii)  $\forall^{< 1} x \perp \equiv \perp$ .
- (iv)  $\forall^{< n} x F \models \forall^{< m} x F$ , if  $n \leq m$ .

In [Beh22a] a dedicated symbolic notation that expresses counting quantifiers for numbers  $n$  with  $n$  stacked arcs is introduced. It covers basic formulas of form (d.) in Theorem 2 and their negations. For example, if  $\alpha[x]$  and  $\beta[x]$  are formulas with free variable  $x$ , then

$$\underbrace{\alpha \beta}_n$$



stands for  $\forall^{<2}x \alpha[x] \vee \neg\beta[x]$  (or, equivalently,  $\neg\exists^{\geq 2}x \neg\alpha[x] \wedge \beta[x]$ ) and

$$\overline{\alpha\beta}$$

for  $\exists^{\geq 2}x \alpha[x] \wedge \neg\beta[x]$ . The role of  $\top$  and  $\perp$  for the empty conjunction and disjunction, respectively, is played in [Beh22a] by symbols  $V$  and  $\Lambda$  for the universal and the empty class, which would correspond to versions of  $\top$  and  $\perp$  that are notated like unary predicates.

As discussed in Sect. 11.2, elimination of *all* predicate quantifiers in a formula without constants, without free individual variables and without free predicate occurrences yields formulas that can be represented by a Boolean combination of basic formulas of just the form  $\exists^{\geq n}x \top$  with  $n \geq 1$ . We call such a Boolean combination a *pure counting formula*. A straightforward way to decide validity of pure counting formulas is by conversion to conjunctive normal form, replacing literals  $\neg\exists^{\geq n}x \top$  with the equivalent  $\forall^{<n}x \perp$  (which we now also accept as basic formulas) and simplifying each clause by Prop. 10.iv and 12.iv such that each clause is either empty (that is, is  $\perp$ ), contains a single basic formula or contains exactly two basic formulas, one with existential and the other with universal counting quantifier. The formula is then valid if and only if each clause is valid, that is, (i) is not empty, or (ii) contains just  $\exists^{\geq 1}x \top$ , or (iii) contains  $\exists^n x \top$  and  $\forall^{<m}x \perp$ , where  $n \leq m$ .

Dually, satisfiability of pure counting formulas can be decided by analogous transformation to disjunctive normal form. The formula is then satisfiable if and only if each (conjunctive) clause is satisfiable, that is, (i) is empty, or (ii) contains just a single basic formula that is different from  $\forall^{<1}x \perp$ , or (iii) contains two basic formulas  $\exists^n x \top$  and  $\forall^{<m}x \perp$ , where  $n < m$ .

Following [Beh22a], this disjunctive form illustrates how domain cardinalities are constrained by a pure counting formula: Each (conjunctive) clause justifies a series of numbers with a lower limit or with lower as well as upper limits as domain cardinalities. A pure counting formula is thus either true for all domain cardinalities with exception of a finite number or false for all domain cardinalities with exception of a finite number. For sufficiently large domains, in particular for all infinite domains, a pure counting formula is then either valid or unsatisfiable.

### 13.2 Conversion to Counting Quantifier Normal Form

We now approach the proof of Theorem 2. It is helpful to isolate the following key step as a lemma on its own:

**Lemma 13 (Auxiliary Elimination Lemma for First-Order Logic with Equality).** *For all first-order formulas  $F[x]$ , where  $x$  is an individual variable that possibly occurs free in  $F[x]$ , for all sequences  $T = \{t_1, \dots, t_n\}$  of  $n \geq 0$  distinct constants or variables which are different from  $x$  and do not occur in  $F[x]$  and for all integers  $m \geq 1$  let  $\text{NDT}_{F[x],T}(m)$  be the formula*

$$\text{NDT}_{F[x],T}(m) \stackrel{\text{def}}{=} \bigwedge_{\substack{S \subseteq T \\ |S|=m}} \left( \bigvee_{t \in S} \neg F[t] \vee \bigvee_{\substack{t_i, t_j \in S \\ i < j}} t_i = t_j \right),$$

where  $F[t]$  denotes  $F[x]$  with all free occurrences of  $x$  replaced by  $t$ . Then

$$\exists x F[x] \wedge \bigwedge_{1 \leq i \leq n} x \neq t_i$$

is equivalent to

$$(i) \quad \bigvee_{1 \leq m \leq n} ((\exists^{\geq m} x F[x]) \wedge \text{NDT}_{F[x],T}(m)) \vee \exists^{\geq n+1} x F[x].$$

$$(ii) \quad (\exists^{\geq 1} x F[x]) \wedge \bigwedge_{1 \leq m \leq n} ((\exists^{\geq m+1} x F[x]) \vee \text{NDT}_{F[x],T}(m)).$$

The shorthand  $\text{NDT}_{F[x],T}(m)$  in Lemma 13 (suggesting *No m are Denoted by a Term*) expresses that it is false that  $F$  applies to  $m$  different individuals denoted by terms  $t \in T$ , which is immediate from the following alternate way to write its definition:

$$\text{NDT}_{F[x],T}(m) \equiv \bigwedge_{\substack{S \subseteq T \\ |S|=m}} \neg \left( \bigwedge_{t \in S} F[t] \wedge \bigwedge_{\substack{t_j, t_k \in S \\ j < k}} t_j \neq t_k \right). \quad (3)$$

Propositions 13.i and 13.ii show a disjunctive form and a conjunctive form in which the existential first-order quantifier in  $\exists x F[x] \wedge \bigwedge_{1 \leq i \leq n} x \neq t_i$  can be “eliminated” in favor of counting quantifiers whose scopes do not include the disequalities  $x \neq t_i$ . One form can be obtained from the other by distributing connectives and simplifying. As remarked in [Beh22a], Lemma 13 can be proven on the basis that the left side of the proposition, that is,  $\exists x F[x] \wedge \bigwedge_{1 \leq i \leq n} x \neq t_i$ , is implied by each disjunct of the right side of the disjunctive form (Lemma 13.i) and implies each conjunct of the right side of the conjunctive form (Lemma 13.ii).

Based on Lemma 13, we now prove Theorem 2, the main theorem of “quantifier elimination” for monadic first-order formulas with equality, following roughly the template from [CK90, Sect. 1.5]. As explained in Sect. 10.4, we explicitly assert the *existence of an effective method* in the theorem.

**Theorem 2 (Counting Quantifier Normal Form for  $\text{MON}_=$ ).** *There is an effective method to compute from a given  $\text{MON}_=$  formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a Boolean combination of basic formulas of the form:
  - (a)  $p$ , where  $p$  is a nullary predicate,
  - (b)  $pt$ , where  $p$  is a unary predicate and  $t$  is a constant or an individual variable,
  - (c)  $t = s$ , where each of  $t, s$  is a constant or an individual variable,
  - (d)  $\exists^{\geq n} x \bigwedge_{1 \leq i \leq m} L_i[x]$ , where  $n \geq 1$ ,  $m \geq 0$  and the  $L_i[x]$  are pairwise different and pairwise non-complementary positive or negative literals with a unary predicate applied to the individual variable  $x$ ,
2.  $F' \equiv F$ ,
3. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* Existence of a method that meets items (1.) and (2.) of the theorem statement follows if (i) every atomic formula of the input class is a basic formula, and (ii) there is an effective method to compute for all formulas of the form  $\exists v F$  where  $v$  is an individual variable and  $F$  a Boolean combination of basic formulas an equivalent Boolean combination of basic formulas. The overall method to compute for a given formula in the input class an equivalent Boolean combination of basic formulas then consists in exhaustively rewriting (as explained in the proof of Corollary 4) subformula occurrences of the form of case (ii) to a Boolean combination of basic formulas.

Property (i) is easy to see: Every atomic MON<sub>=</sub> formula is clearly a basic formula of the specified form (a.), (b.) or (c.).

We now prove (ii): Let  $F$  be a Boolean combination of basic formulas and let  $v$  be a variable. We show how the formula  $\exists v F$  can be converted with equivalence preserving formula rewritings to a Boolean combination of basic formulas. Unless indicated otherwise, the referenced equivalences are applied there from left to right.

1. Convert  $F$  to a disjunction  $F_1 \vee \dots \vee F_n$ , where  $n \geq 0$ , of conjunctions of basic formulas or negated basic formulas (EQ 1–EQ 3, EQ 19). The result is a generalization of disjunctive normal form, where basic formulas play the role of atoms.
2. Distribute the existential quantifier over the disjunction to obtain  $\exists v F_1 \vee \dots \vee \exists v F_n$ , which is equivalent to  $\exists v F$  (EQ 22 right to left).
3. Convert each disjunct  $\exists v F_i$ , for  $1 \leq i \leq n$ , separately to a Boolean combination  $F'_i$  of basic formulas and disjoin the results. The resulting formula  $F'_1 \vee \dots \vee F'_n$  is a Boolean combination of basic formulas which is equivalent to  $\exists v F$ . The following equivalence preserving transformation steps are applied to each disjunct  $\exists v F_i$ :
  - 3.1. In case there are two complementary conjuncts return with  $F'_i = \perp$  (EQ 6–EQ 7, EQ 17, EQ 12, EQ 16).
  - 3.2. In case  $t \neq t$  is a conjunct, where  $t$  is a term, return with  $F'_i = \perp$  (EQ 12, EQ 16).
  - 3.3. Remove conjuncts of the form  $t = t$ , where  $t$  is a term (EQ 11).
  - 3.4. Orient conjuncts  $t = v$  and  $t \neq v$ , where  $t$  is a term (different from  $v$ , as ensured by the previous steps) to the forms  $v = t$  and  $v \neq t$ , respectively.
  - 3.5. Reorder the conjuncts, remove duplicate conjuncts, and propagate the existential quantifier inward such that exactly those conjuncts are in its scope in which  $v$  does occur as a free variable (EQ 6–EQ 8, EQ 23 from right to left). The resulting formula then has the form

$$G \wedge \exists v \bigwedge_{1 \leq i \leq k} L_i[v] \wedge \bigwedge_{1 \leq i \leq l} v \neq t_i \wedge \bigwedge_{1 \leq i \leq m} v = u_i,$$

where  $k, l, m \geq 0$ , the  $L_i[v]$  are pairwise different positive or negative literals with an unary predicate applied to  $v$  and the  $t_i$  and  $u_i$  are terms. In particular, the  $t_i$  are pairwise different and also different from  $v$ . The subformula  $G$  contains the conjuncts with no free occurrences of  $v$ .

3.6. In case  $k = l = m = 0$ , return with  $F'_i = G$ .

3.7. In case  $m > 0$ , return with

$$F'_i = G \wedge \bigwedge_{1 \leq i \leq k} L_i[u_1] \wedge \bigwedge_{1 \leq i \leq n} u_1 \neq t_i \wedge \bigwedge_{1 < i \leq m} u_1 = u_i,$$

where the  $L_i[u_1]$  are obtained from the literals  $L_i[v]$  by replacing their argument variable  $v$  with  $u_1$ . This transformation step is justified by applying EQ 33 from right to left.

3.8. Finally, in the remaining case  $m = 0$ , return with

$$F'_i = G \wedge G',$$

where  $G'$  is a Boolean combination of basic formulas that is equivalent to

$$\exists v \bigwedge_{1 \leq i \leq k} L_i[v] \wedge \bigwedge_{1 \leq i \leq l} v \neq t_i,$$

as obtained according to either Lemma 13.i or 13.ii.

Item (3.) of the theorem statement, that is, all free individual variables, constants and predicates in  $F'$  do occur in  $F$ , follows since in the transformations involved to compute  $F'$  at no point bound variables are moved outside the scope of their binding quantifier, the only variables introduced in the transformations are bound by counting quantifiers, and neither constants nor predicates are newly introduced.  $\square$

We conclude this subsection with some examples that illustrate the conversion of equality literals to counting quantifiers performed by the method of Theorem 2. In the following example the input formula contains positive equality atoms:

$$\begin{aligned} & \exists x px \wedge x \neq a \wedge x \neq b \wedge x = c \wedge x = d \\ \equiv & pc \wedge c \neq a \wedge c \neq b \wedge c = d. \end{aligned} \quad (4)$$

The following two sequences of equivalences show results obtained if NDT is expanded according to Lemma 13.i:

$$\begin{aligned} & \exists x px \wedge x \neq a \\ \equiv & ((\exists^{\geq 1} x px) \wedge \text{NDT}_{px, \{a\}}(1)) \vee \exists^{\geq 2} x px \\ \equiv & ((\exists^{\geq 1} x px) \wedge \neg pa) \vee \exists^{\geq 2} x px. \end{aligned} \quad (5)$$

$$\begin{aligned} & \exists x px \wedge x \neq a \wedge x \neq b \\ \equiv & ((\exists^{\geq 1} x px) \wedge \text{NDT}_{px, \{a, b\}}(1)) \vee \\ & ((\exists^{\geq 2} x px) \wedge \text{NDT}_{px, \{a, b\}}(2)) \vee \\ & \exists^{\geq 3} x px \\ \equiv & ((\exists^{\geq 1} x px) \wedge \neg pa \wedge \neg pb) \vee \\ & ((\exists^{\geq 2} x px) \wedge (\neg pa \vee \neg pb \vee a = b)) \vee \\ & \exists^{\geq 3} x px. \end{aligned} \quad (6)$$

For the same input formulas, if NDT is expanded according to Lemma 13.ii, then we obtain the following results:

$$\begin{aligned}
 & \exists x px \wedge x \neq a \\
 \equiv & ((\exists^{\geq 1} x px) \wedge ((\exists^{\geq 2} x px) \vee \text{NDT}_{px, \{a\}}(1))) \\
 \equiv & ((\exists^{\geq 1} x px) \wedge ((\exists^{\geq 2} x px) \vee \neg pa)).
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 & \exists x px \wedge x \neq a \wedge x \neq b \\
 \equiv & \exists^{\geq 1} x px && \wedge \\
 & ((\exists^{\geq 2} x px) \vee \text{NDT}_{px, \{a, b\}}(1)) && \wedge \\
 & ((\exists^{\geq 3} x px) \vee \text{NDT}_{px, \{a, b\}}(2)) && \\
 \equiv & \exists^{\geq 1} x px && \wedge \\
 & ((\exists^{\geq 2} x px) \vee (\neg pa \wedge \neg pb)) && \wedge \\
 & ((\exists^{\geq 3} x px) \vee \neg pa \vee \neg pb \vee a = b).
 \end{aligned} \tag{8}$$

## 14 Predicate Elimination for $\text{MON}_=$

In this section Theorem 1, the core result of [Beh22a], is proven. It is first shown that an existential predicate quantification of a formula in the counting quantifier normal form (as produced by the method of Theorem 2) can be converted such that all occurrences of predicate quantification are in subformulas of a particular form, which we call here *Generalized Eliminationshauptform* because it generalizes the *Eliminationshauptform* ((2) on p. 21), by the use of counting quantifiers instead of standard quantifiers. Like the proper *Eliminationshauptform*, the *Generalized Eliminationshauptform* can be converted to an equivalent formula that allows to eliminate the second-order quantification upon  $p$  with Lemma 3.

It is possible to convert the *Generalized Eliminationshauptform* first to a formula with a subformula of the proper *Eliminationshauptform* and then perform elimination just based on the latter. However, following [Beh22a], elimination is shown here directly for the *Generalized Eliminationshauptform*. The proof actually involves a conversion such that the predicate quantification has the form of the special case of the proper *Eliminationshauptform* where the constituents  $\bigwedge_{1 \leq i \leq c} (\exists x C_i[x] \wedge px)$  and  $\bigwedge_{1 \leq i \leq d} (\exists x D_i[x] \wedge \neg px)$  are “empty”, that is,  $c = d = 0$ , which allows to perform elimination directly with Lemma 3. Although technically covered by the general case with equality, we will discuss the simpler handling of the proper *Eliminationshauptform* in the next section. The proof of Theorem 1 is split up into several lemmas. First, the construction of the *Generalized Eliminationshauptform* is shown.

**Lemma 14 (Constructing the *Generalized Eliminationshauptform* for  $\text{MON}_=$  Formulas).** *There is an effective method to compute for a given unary predicate  $p$  and  $\text{MON}_=$  formula  $F$  formula  $F'$  such that*

1.  $F'$  is a  $\text{QMON}_=$  formula,
2.  $F' \equiv \exists p F$ ,

3.  $p$  is the only quantified predicate in  $F'$ ,
4. All occurrences of  $p$  in  $F'$  are in subformulas of the form

$$\begin{aligned} & \exists p \bigwedge_{1 \leq i \leq a} (\forall x^{<a_i} A_i[x] \vee px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq b} (\forall x^{<b_i} B_i[x] \vee \neg px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} (\exists x^{\geq c_i} C_i[x] \wedge px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} (\exists x^{\geq d_i} D_i[x] \wedge \neg px), \end{aligned}$$

where  $a, b, c, d$  are natural numbers  $\geq 0$  and for the referenced values of  $i$  the  $a_i, b_i, c_i, d_i$  are natural numbers  $\geq 1$ , and the  $A_i[x], B_i[x], C_i[x], D_i[x]$  are first-order formulas in which  $p$  does not occur,

5. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* The result formula  $F'$  is obtained from  $\exists p F$  by a sequence of transformations that preserve equivalence and do neither introduce fresh constants nor predicates nor variables without a binding quantifier. First, transform  $F$  according to Theorem 2 to counting quantifier normal form, that is, to a Boolean combination of basic formulas in the sense of Theorem 2. Convert the result to a disjunction of conjunctions of such basic formulas or negated such basic formulas (like a disjunctive normal form, but with basic formulas in the role of atoms). Remove disjuncts that contain complementary conjuncts (EQ 29).

Let  $F''$  be this intermediate result, which is equivalent to  $F$ . Proceed with transforming  $\exists p F''$ . Distribute the existential quantifier upon  $p$  over the disjuncts (EQ 22 right to left), remove it from disjuncts in which  $p$  does not occur (EQ 24), and propagate it inward into the disjuncts (EQ 6–EQ 8, EQ 23 right to left) such that their argument is a conjunction of basic formulas. Let  $F'''$  be the intermediate result obtained so far.

In  $F'''$ , replace all subformulas of the form  $\neg \exists^{\geq n} x \bigwedge_{1 \leq i \leq m} L_i[x]$  with the equivalent  $\forall^{<n} x \bigvee_{1 \leq i \leq m} \overline{L_i[x]}$ . Replace all subformulas of the form  $\neg pt$  where  $t$  is a constant or an individual variable that is free in  $F$  with the equivalent  $\forall^{<1} x x \neq t \vee \neg px$  (EQ 32, 12.ii). Replace all subformulas of the form  $pt$  where  $t$  is a constant or an individual variable that is free in  $F$  with the equivalent  $\forall^{<1} x x \neq t \vee px$ . Reorder disjuncts and conjuncts in the scope of  $\exists p$  such that form (4) in the lemma statement is matched (EQ 6–EQ 7). The formula has now all properties asserted about  $F'$  by the lemma to prove.  $\square$

If the quantified predicate is nullary, the first steps of the method of Lemma 14 already yield a particularly simple form:

**Lemma 15 (Normalizing Quantification Upon Nullary Predicates).**

*There is an effective method to compute for a given nullary predicate  $p$  and MON<sub>=</sub> formula  $F$  formula  $F'$  such that*

1.  $F'$  is a QMON<sub>=</sub> formula,
2.  $F' \equiv \exists p F$ ,
3.  $p$  is the only quantified predicate in  $F'$ ,
4. All occurrences of  $p$  in  $F'$  are in subformulas of the forms  $\exists p p$  and  $\exists p \neg p$ ,
5. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* The intermediate result  $F'''$  computed described in the proof of Lemma 14 has all properties asserted about  $F'$  by the lemma to prove.  $\square$

The following Lemma shows for quantification upon a unary predicate the conversion of the *Generalized Eliminationshauptform* to a form that allows to perform elimination directly by application of Lemma 3.

**Lemma 16 (From Generalized Eliminationshauptform to the Basic Elimination Lemma).** *Let  $p$  be a unary predicate and let  $F$  be a formula of the form*

$$\begin{aligned} & \exists p \bigwedge_{1 \leq i \leq a} (\forall x^{<a_i} A_i[x] \vee px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq b} (\forall x^{<b_i} B_i[x] \vee \neg px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} (\exists x^{\geq c_i} C_i[x] \wedge px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} (\exists x^{\geq d_i} D_i[x] \wedge \neg px), \end{aligned}$$

where  $a, b, c, d$  are natural numbers  $\geq 0$ , for the referenced values of  $i$  the  $a_i, b_i, c_i, d_i$  are natural numbers  $\geq 1$ , and the  $A_i[x], B_i[x], C_i[x], D_i[x]$  are first-order formulas in which  $p$  does not occur. Then  $F$  is equivalent to

$$Q G \wedge \exists p (\forall x A[x] \vee px) \wedge (\forall x B[x] \vee \neg px),$$

where  $Q$  is a quantifier prefix that existentially quantifies upon the following individual variables which are fresh, that is, do not occur in  $F$ :

$$\begin{aligned} & x_{i1} \dots x_{i(a_i-1)}, \quad \text{for } 1 \leq i \leq a \\ & y_{i1} \dots y_{i(b_i-1)}, \quad \text{for } 1 \leq i \leq b, \\ & u_{i1} \dots u_{i c_i}, \quad \text{for } 1 \leq i \leq c, \\ & v_{i1} \dots v_{i d_i}, \quad \text{for } 1 \leq i \leq d, \end{aligned}$$

where  $G$  is the formula

$$\begin{aligned} & \bigwedge_{1 \leq i \leq c, 1 \leq j \leq c_i} (C_i[u_{ij}] \wedge \bigwedge_{j < k \leq c_i} u_{ij} \neq u_{ik}) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d, 1 \leq j \leq d_i} (D_i[v_{ij}] \wedge \bigwedge_{j < k \leq d_i} v_{ij} \neq v_{ik}), \end{aligned}$$

with  $C_i[u_{ij}]$  and  $D_i[v_{ij}]$  denoting  $C_i[x]$  and  $D_i[x]$  after replacing all free occurrences of  $x$  by  $u_{ij}$  and  $v_{ij}$ , respectively,

where  $A[x]$  is the formula

$$\bigwedge_{1 \leq i \leq a} (A_i[x] \vee \bigvee_{1 \leq j < a_i} x = x_{ij}) \wedge \bigwedge_{1 \leq i \leq c, 1 \leq j \leq c_i} x \neq u_{ij},$$

and where  $B[x]$  is the formula

$$\bigwedge_{1 \leq i \leq b} (B_i[x] \vee \bigvee_{1 \leq j < b_i} x = y_{ij}) \wedge \bigwedge_{1 \leq i \leq d, 1 \leq j \leq d_i} x \neq v_{ij}.$$

*Proof.* Universal and existential counting quantifiers in  $F$  can be expanded in alternate ways (Prop. 11.ii and 9.i) such that in both cases existential variables are produced which can be moved (by EQ 23, EQ 25) in front of the existential predicate quantifier. The particular expansions of universal counting quantifiers applied there are:

$$\begin{aligned}
& \forall x^{\overline{a_i}} A_i[x] \vee px \\
\equiv & \exists x_{i1} \dots \exists x_{i(a_i-1)} \forall x A_i[x] \vee px \vee \bigvee_{1 \leq j < a_i} x = x_{ij} && \text{by Prop. 11.ii} \\
\equiv & \exists x_{i1} \dots \exists x_{i(a_i-1)} \forall x (A_i[x] \vee \bigvee_{1 \leq j < a_i} x = x_{ij}) \vee px, && \text{by EQ 6, EQ 7}
\end{aligned}$$

and analogously

$$\begin{aligned}
& \forall x^{\overline{b_i}} B_i[x] \vee \neg px \\
\equiv & \exists y_{i1} \dots \exists y_{i(b_i-1)} \forall x (B_i[x] \vee \bigvee_{1 \leq j < b_i} x = y_{ij}) \vee \neg px.
\end{aligned}$$

The involved expansions of existential counting quantifiers are:

$$\begin{aligned}
& \exists^{\geq c} x C_i[x] \wedge px \\
\equiv & \exists u_{i1} \dots \exists u_{ic} \bigwedge_{1 \leq j \leq c} (C_i[u_{ij}] \wedge pu_{ij} \wedge \bigwedge_{j < k \leq c} u_{ij} \neq u_{ik}) && \text{by Prop. 9.i} \\
\equiv & \exists u_{i1} \dots \exists u_{ic} \bigwedge_{1 \leq j \leq c} (C_i[u_{ij}] \wedge \bigwedge_{j < k \leq c} u_{ij} \neq u_{ik}) \wedge && \text{by EQ 6, EQ 7} \\
& \quad \bigwedge_{1 \leq j \leq c} pu_{ij} \\
\equiv & \exists u_{i1} \dots \exists u_{ic} \bigwedge_{1 \leq j \leq c} (C_i[u_{ij}] \wedge \bigwedge_{j < k \leq c} u_{ij} \neq u_{ik}) \wedge && \text{by EQ 32} \\
& \quad \bigwedge_{1 \leq j \leq c} (\forall x x \neq u_{ij} \vee px),
\end{aligned}$$

and analogously

$$\begin{aligned}
& \exists^{\geq d_i} x D_i[x] \wedge \neg px \\
\equiv & \exists v'_{i1} \dots \exists v'_{id_i} \bigwedge_{1 \leq j \leq d_i} (D_i[v'_{ij}] \wedge \bigwedge_{j < k \leq d_i} v'_{ij} \neq v'_{ik}) \wedge \\
& \quad \bigwedge_{1 \leq j \leq d_i} (\forall x x \neq v'_{ij} \vee \neg px).
\end{aligned}$$

The formulas  $A[x]$  and  $B[x]$  are obtained from the universally quantified conjuncts in the shown expansions by merging universal quantifiers (EQ 21) and factoring the disjuncts  $px$  and  $\neg px$ , respectively (EQ 20 left to right).  $\square$

We can now combine the lemmas of this section to the proof of Theorem 1.

**Theorem 1 (Predicate Elimination for  $\text{MON}_=$ ).** *There is an effective method to compute from a given predicate  $p$  and  $\text{MON}_=$  formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a  $\text{MON}_=$  formula,
2.  $F' \equiv \exists p F$ ,
3.  $p$  does not occur in  $F'$ ,
4. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* If  $p$  is nullary, then apply the method according to Lemma 15 to  $p$  and  $F$ , followed by replacing all occurrences of  $\exists p p$  and of  $\exists p \neg p$  with the equivalent  $\top$ .

In case  $p$  is unary, apply the method according to Lemma 14 to  $p$  and  $F$ , followed by replacing all subformulas starting with  $\exists p$  with the equivalent formulas according to Lemma 16. In the intermediate result, replace all subformulas starting with  $\exists p$  with the equivalent formulas obtained by eliminating  $\exists p$  according to Lemma 3.  $\square$



## 15 Predicate Elimination for Formulas without Equality

Theorem 1 and the material leading to that theorem applies to formulas *with equality*. In [Beh22a], a simpler variant that only applies to formulas *without equality* is described first and in more detail, along with a direct comparison to an earlier result by Schröder and with a restructuring of the result as a postprocessing operation after elimination. In this section we adapt the essential steps of this variant and discuss aspects that become apparent more clearly with that simpler method.

As already mentioned in Sect. 11.1, even if the input formula is without equality, the steps involved in predicate quantifier elimination can possibly introduce equality if the predicate to be eliminated occurs with an argument that is not a universally quantified variable. Thus, if there are several predicates to be eliminated one by one, an input formula without equality can only be safely assumed for elimination of the first predicate.

However, as can be seen from the proofs in this section, the equality literals introduced for inputs without equality actually either have a constant or two existential variables as arguments, implying that the simpler variant without dedicated equality handling is sufficient for elimination in formulas  $\exists p_1 \dots \exists p_n F$  where  $F$  is a MON formula. For this case, and the dual elimination of a sequence of universal predicate quantifiers, Behmann [Beh22a] sketches a generalization of the variant for formulas without equality, which, however, involves a translation whose size is exponential in the number of predicate quantifiers. The core idea for Behmann's generalization is described below at the end of this section.

The following proposition shows a method to compute a normal form for monadic first-order logic *without equality*, which is like the counting quantifier normal form of Theorem 2, but with certain further restrictions on the allowed basic formulas. In particular, it involves no counting quantifiers.

**Proposition 17 (Normal Form for MON).** *There is an effective method to compute for a given MON formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a Boolean combination of basic formulas of the form:
  - (a)  $p$ , where  $p$  is a nullary predicate,
  - (b)  $pt$ , where  $p$  is a unary predicate and  $t$  is a constant or individual variable,
  - (c)  $\exists x \bigwedge_{1 \leq i \leq m} L_i[x]$ , where the  $L_i[x]$  are pairwise different and pairwise non-complementary positive or negative literals with a unary predicate applied to the individual variable  $x$ ,
2.  $F' \equiv F$ ,
3. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* If the proof of Theorem 2 is instantiated with a formula  $F$  without equality, the resulting formula is a Boolean combination of basic formulas of the forms (a), (b) as shown above and of the form (d) from Theorem 2 with  $n$  instantiated to 1. The latter formulas can be converted to the form (c) shown above by replacing the counting quantifier  $\exists^{\geq 1}x$  with the equivalent standard quantifier  $\exists x$  (Prop. 10.ii).  $\square$

The remaining two propositions and lemmas in this section give, based on the normal form of Prop. 17, the ingredients for a method to eliminate predicate quantification on MON formulas, in analogy to Lemma 14 and 15. The construction of the *Eliminationshauptform* from arbitrary such formulas is ensured with the following lemma.

**Lemma 18 (Constructing the Eliminationshauptform for MON).** *There is an effective method to compute for a given unary predicate  $p$  and MON formula  $F$  a formula  $F'$  such that*

1.  $F'$  is a  $\text{MON}_=$  formula,
2.  $F' \equiv \exists p F$ ,
3.  $p$  is the only quantified predicate in  $F'$ ,
4. All occurrences of  $p$  in  $F'$  are in subformulas of the form

$$\begin{aligned} \exists p \bigwedge_{1 \leq i \leq a} (\forall x A_i[x] \vee px) \quad \wedge \\ \bigwedge_{1 \leq i \leq b} (\forall x B_i[x] \vee \neg px) \quad \wedge \\ \bigwedge_{1 \leq i \leq c} (\exists x C_i[x] \wedge px) \quad \wedge \\ \bigwedge_{1 \leq i \leq d} (\exists x D_i[x] \wedge \neg px), \end{aligned}$$

where  $a, b, c, d$  are natural numbers  $\geq 0$  and the  $A_i[x]$ ,  $B_i[x]$ ,  $C_i[x]$ ,  $D_i[x]$  are formulas in which  $p$  does not occur,

5. All free individual variables, constants and predicates in  $F'$  do occur in  $F$ .

*Proof.* Compute the intermediate  $F'''$  as described in the proof of Lemma 14. Continue in analogy to that proof: In  $F'''$ , replace all subformulas of the form  $\neg \exists x \bigwedge_{1 \leq i \leq m} L_i[x]$  with the equivalent  $\forall x \bigvee_{1 \leq i \leq m} \overline{L}_i[x]$ . Replace all subformulas of the form  $\neg pt$  where  $t$  is a constant or an individual variable that is free in  $F$  with the equivalent  $\forall x x \neq t \vee \neg px$ . Replace all subformulas of the form  $pt$  where  $t$  is a constant or an individual variable that is free in  $F$  with the equivalent  $\forall x x \neq t \vee px$ . Reorder disjuncts and conjuncts in the scope of  $\exists p$  such that form (4) in the lemma statement is matched. The formula has now all properties asserted about  $F'$  by the lemma to prove.  $\square$

The method described in the proof of Lemma 18 introduces equality atoms in case there are occurrences of the quantified predicate where the argument is a constant or an individual variable that is free in the original input formula.

The following Lemma renders the elimination result as given in [Beh22a]. It combines conversion from *Eliminationshauptform* to a form that allows direct elimination with Lemma 3, in analogy to Lemma 16, with performing the elimination and restructuring the result.

**Lemma 19 (Elimination on the Eliminationshauptform).** *Let  $F$  be a formula of the following form:*

$$\begin{aligned} \exists p \bigwedge_{1 \leq i \leq a} (\forall x A_i[x] \vee px) \quad \wedge \\ \bigwedge_{1 \leq i \leq b} (\forall x B_i[x] \vee \neg px) \quad \wedge \\ \bigwedge_{1 \leq i \leq c} (\exists x C_i[x] \wedge px) \quad \wedge \\ \bigwedge_{1 \leq i \leq d} (\exists x D_i[x] \wedge \neg px), \end{aligned}$$

where  $p$  is a unary predicate,  $a, b, c, d$  are numbers  $\geq 0$  and the  $A_i[x]$ ,  $B_i[x]$ ,  $C_i[x]$ ,  $D_i[x]$  are first-order formulas in which  $p$  does not occur. Then  $F$  is equivalent to the following formula  $\text{MON}_=$  formula  $F'$ :

$$\begin{aligned} & (\forall x \bigwedge_{1 \leq i \leq a} A_i[x] \vee \bigwedge_{1 \leq i \leq b} B_i[x]) \wedge \\ & \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \\ & \bigwedge_{1 \leq i \leq c, 1 \leq j \leq d} u_i \neq v_j \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} (C_i[u_i] \wedge \bigwedge_{1 \leq j \leq b} B_j[u_i]) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} (D_i[v_i] \wedge \bigwedge_{1 \leq j \leq a} A_j[v_i]), \end{aligned}$$

where  $u_1, \dots, u_c$  and  $v_1, \dots, v_d$  are distinct individual variables that are fresh, that is, do not occur free in  $F$ , and if  $t$  is one of these variables, then  $A_i[t]$ ,  $B_i[t]$ ,  $C_i[t]$ ,  $D_i[t]$  denote  $A_i[x]$ ,  $B_i[x]$ ,  $C_i[x]$ ,  $D_i[x]$ , respectively, with all free occurrences of  $x$  replaced by  $t$ .

*Proof.* The proof shows equivalence preserving transformations that lead from  $F$  to  $F'$ : First,  $F$  is converted to a form that matches the left side of Lemma 3. That lemma is then applied to eliminate the existentially quantified  $p$ . Finally, the formula that results from applying the lemma is postprocessed to yield  $F'$ .

We define two shorthands  $A[x] = \bigwedge_{1 \leq i \leq a} A_i[x]$  and  $B[x] = \bigwedge_{1 \leq i \leq b} B_i[x]$ . Then  $F$  is equivalent to

$$\begin{aligned} & \exists p \forall x A[x] \vee px \quad \wedge \\ & \forall x B[x] \vee \neg px \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} (\exists x C_i[x] \wedge px) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} (\exists x D_i[x] \wedge \neg px). \end{aligned}$$

By renaming the existential individual variables (EQ 26), moving the existential quantifiers outward (EQ 23) and reordering them (EQ 25) we obtain:

$$\begin{aligned} & \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \exists p \forall x A[x] \vee px \quad \wedge \\ & \forall x B[x] \vee \neg px \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} (C_i[u_i] \wedge pu_i) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} (D_i[v_i] \wedge \neg pv_i). \end{aligned}$$

Reordering conjuncts (EQ 7, EQ 6) and moving the quantifier upon  $p$  inward (EQ 23 from right to left) yields:

$$\begin{aligned} & \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \bigwedge_{1 \leq i \leq c} C_i[u_i] \wedge \bigwedge_{1 \leq i \leq d} D_i[v_i] \quad \wedge \\ & \exists p \forall x A[x] \vee px \quad \wedge \\ & \forall x B[x] \vee \neg px \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} pu_i \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} \neg pv_i. \end{aligned}$$

Next “pull out” the existentially quantified arguments from  $p$  (EQ 32):

$$\begin{aligned} \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \bigwedge_{1 \leq i \leq c} C_i[u_i] \wedge \bigwedge_{1 \leq i \leq d} D_i[v_i] \wedge \\ \exists p \forall x A[x] \vee px \quad \wedge \\ \forall x B[x] \vee \neg px \quad \wedge \\ \bigwedge_{1 \leq i \leq c} (\forall x x \neq u_i \vee px) \quad \wedge \\ \bigwedge_{1 \leq i \leq d} (\forall x x \neq v_i \vee \neg px). \end{aligned}$$

By merging universal quantifiers (EQ 21) and factoring the occurrences of  $px$  and  $\neg px$  (EQ 20 from right to left) we get:

$$\begin{aligned} \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \bigwedge_{1 \leq i \leq c} C_i[u_i] \wedge \bigwedge_{1 \leq i \leq d} D_i[v_i] \quad \wedge \\ \exists p \forall x ((A[x] \wedge \bigwedge_{1 \leq i \leq c} x \neq u_i) \vee px) \quad \wedge \\ \forall x ((B[x] \wedge \bigwedge_{1 \leq i \leq d} x \neq v_i) \vee \neg px). \end{aligned}$$

The subformula starting with  $\exists p$  matches the left side of Lemma 3, which justifies to rewrite it by a first-order formula that does no longer contain  $p$ , resulting in:

$$\begin{aligned} \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \bigwedge_{1 \leq i \leq c} C_i[u_i] \wedge \bigwedge_{1 \leq i \leq d} D_i[v_i] \wedge \\ \forall x (A[x] \wedge \bigwedge_{1 \leq i \leq c} x \neq u_i) \vee \\ (B[x] \wedge \bigwedge_{1 \leq i \leq d} x \neq v_i). \end{aligned}$$

By distributing disjunction over conjunction (EQ 20), distributing universal quantification into conjunction (EQ 21 from right to left), “pulling in” argument terms (EQ 32 from right to left) and reordering conjuncts (EQ 6, EQ 7) we obtain:

$$\begin{aligned} (\forall x A[x] \vee B[x]) \quad \wedge \\ \exists u_1 \dots \exists u_c \exists v_1 \dots \exists v_d \bigwedge_{1 \leq i \leq c, 1 \leq j \leq d} u_i \neq v_j \quad \wedge \\ \bigwedge_{1 \leq i \leq c} (C_i[u_i] \wedge B[u_i]) \quad \wedge \\ \bigwedge_{1 \leq i \leq d} (D_i[v_i] \wedge A[v_i]), \end{aligned}$$

where, in analogy to the previously used notation,  $A[v_i]$  and  $B[u_i]$  denotes  $A[x]$  and  $B[x]$  with all free occurrences of  $x$  replaced by  $v_i$  or  $u_i$ , respectively. Expanding the shorthands  $A[x]$ ,  $B[x]$ ,  $A[v_i]$ ,  $B[u_i]$  then yields the formula  $F'$  from the proposition statement.  $\square$

The method described in the proof of Lemma 19 introduces equality to handle occurrences of the quantified predicate with an existentially quantified argument. With the method described in Lemma 18 this is the second place where equality is introduced in the overall method to eliminate predicates from formulas without equality.

Behmann remarks [Beh22a, p. 201f] that the first division of the second volume of Schröder’s *Algebra der Logik* [Sch91, p. 400 2] concludes with recommending the problem solved with Lemma 19 as an open issue for future research. That Schröder failed to solve the problem is attributed by Behmann to the insufficient means of representation in the *Algebra der Logik*. In the proof of Lemma 19 after the elimination step a further equivalence preserving transformation is applied,

where occurrences of  $A[x]$  and  $B[x]$  are instantiated with existentially quantified variables. (To the proof of Theorem 1 given above, the analogous step could be added.) This step facilitates to compare the result of Lemma 19 with an earlier incomplete result by Schröder: Behmann [Beh22a, p. 203f] remarks that the following formula, related to the last displayed formula in the proof of Lemma 19, is what Schröder calls „*crude resultant*” („*Resultante aus dem Rohen*“) [Sch91, §41] (see also [Cra08] for a modern discussion of Schröder’s work on elimination):

$$\begin{aligned} & (\forall x A[x] \vee B[x]) \quad \wedge \\ & \bigwedge_{1 \leq i \leq c} \exists x (C_i[x] \wedge B[x]) \quad \wedge \\ & \bigwedge_{1 \leq i \leq d} \exists x (D_i[x] \wedge A[x]). \end{aligned} \tag{9}$$

This formula is obtained from the last displayed formula in the proof by dropping the conjunct  $\bigwedge_{1 \leq i \leq c, 1 \leq j \leq d} u_i \neq v_j$ , which leads to a weaker formula, followed by the equivalence preserving inward propagation of existential quantifiers and renaming of variables.

As mentioned in the beginning of this section, in [Beh22a, § 17] a generalization of the techniques for formulas without equality to the simultaneous elimination of a sequence of predicate quantifiers that are all existential or all universal is outlined. It can be based on the following adaption of Lemma 3 to the simultaneous elimination of  $n$  existentially quantified unary predicates from a conjunction of  $2^n$  disjunctions:

$$\exists p_1 \dots \exists p_n \bigwedge_{S \subseteq \{1, \dots, n\}} (\forall x F_S \vee \bigvee_{i \in S} p_i x \vee \bigvee_{i \in \{1, \dots, n\} - S} \neg p_i x) \equiv \forall x \bigvee_{S \subseteq \{1, \dots, n\}} F_S, \tag{10}$$

which holds if the predicates  $p_1, \dots, p_n$  do not occur in the formulas  $F_S$  (subsets  $S$  of  $\{1, \dots, n\}$  are used in  $F_S$  as index subscript).



## Part III

# Further Issues Addressed in and Related to Behmann's Habilitation Thesis





## 16 Introduction to Part III

In this part various issues that are addressed by Behmann in [Beh22a] aside of the main results as well as specific connections with related techniques are discussed. For a further discussion of Behmann’s elimination technique in relation to modern second-order quantifier elimination methods following the direct or Ackermann approach we refer to [Wer15].

## 17 Innex and Related Forms

Recall that the key technique of [Beh22a] is propagating quantifiers inward, also for the price of expensive operations such as distribution of conjunction over disjunction and vice versa.<sup>16</sup> This inward propagation is applied to quantifiers upon individual variables (Theorem 2, Prop. 17) as well as to quantifiers upon predicates (Lemma 14, 15 18).

As remarked in [Beh22a, p. 193], just propagating Boolean quantifiers inward in this manner already yields a decision method for quantified Boolean formulas that have no free predicates (that is, in other terminology, no free Boolean variables), and hence also for propositional logic. This can be easily seen from Corollary 4: If the method asserted in the corollary is applied to a quantified Boolean formula, by the underlying lemmas, in particular Lemma 15, inward propagation of  $\exists p$  yields a formula in which all occurrences of Boolean quantification are in formulas of the form  $\exists p p$  and  $\exists p \neg p$ .

Given the decidability of the Bernays-Schönfinkel class [BS28] (published six years after [Beh22a]), monadic first-order logic can be decided essentially just by propagating all individual quantifiers inward in the manner indicated above with possibly expensive distribution operations. From the result formula, a prenex formula in the Bernays-Schönfinkel class can then be obtained by first propagating existential quantifiers outward and then the universal quantifiers. This conversion has been sketched in [DG79, p. 36].

Techniques to propagate quantifiers upon individual variables inward are known for a long time in automated theorem proving, under the names *miniscope form* and *antiprenexing*. They are typically considered as preprocessing operations [Lov78; NW01], without taking potentially expensive steps like distribution of connectives into account. Exceptions are early works by [Wan60] – according to [Ern71, footnote 2]<sup>17</sup> – and [Bib74], where the possibility to take advantage of distribution is mentioned in the conclusion. The commonly considered preprocessing methods might terminate inward propagation with a formula

<sup>16</sup> On formulas that are not in negation normal form, distribution of conjunction over disjunction alone or distribution of disjunction over conjunction alone is sufficient, because either one can then express the other, consider for example:  $F \vee (G \wedge H) \equiv \neg(\neg F \wedge (\neg G \vee \neg H)) \equiv \neg((\neg F \wedge \neg G) \vee (\neg F \wedge \neg H)) \equiv (F \vee G) \wedge (F \vee H)$ .

<sup>17</sup> The definition of *miniscope form* in [Wan60], however, does not imply this and would accept, for example,  $\exists x px \wedge (qx \vee rx)$ .

where a potentially expensive step would allow further inward propagation. As a consequence, these methods are in general not sufficient to decide MON.

Two variants to propagate individual quantifiers inward have been discussed in the literature. To describe them, we assume quantifiers  $\exists$  and  $\forall$  as they appear in a formula in negation normal form, obtained by propagating negation inward with (EQ 1–EQ 5) such that negated existential quantification is expressed by universal quantification. In the first variant, often called *miniscope form*, existential as well as universal quantifiers are both propagated inward [Wan60; Bib74; Lov78]. The main motivation is there to reduce the number of arguments of Skolem functions by reducing the number of universal variables that have a given existential quantifier in their scope. In the second variant, termed *antiprenexing* in [Egl94],<sup>18</sup> both types of quantifiers are handled differently: only universal quantifiers are propagated inward, existential quantifiers are propagated outward (EQ 22, EQ 23). Aside of reducing the number of arguments of Skolem functions, the motivation is there to reduce also the number of Skolem functions: Applying EQ 22 from right to left would effect duplication of existential quantifiers, each requiring a different Skolem function.

As observed in [Beh22a], the elimination technique for monadic formulas of Lemma 16 and 19 involves propagating predicate quantifiers and universal individual quantifiers *inward*, while propagating existential individual quantifiers *outward*. A systematic investigation of quantifier propagation schemes with respect to elimination of predicate quantifiers seems still an open issue.

## 18 Quine’s Expansion

In [Qui45] Quine presents in a decision method for MON that is, as remarked in [Chu56, pp. 253, 293], a variant of Behmann’s method. Like the latter, it is based on producing the normal form of Prop. 17 by propagating quantifiers upon instance variables inward. However, instead of performing distribution of conjunction over disjunction to enable inward propagation of quantifiers, rewriting with the following equivalence Prop. 20.ii is applied. This equivalence is shown first in a dual version as Prop. 20.i that corresponds to the setting of Prop. 17.

**Proposition 20 (Quine’s Expansion).** *Let  $F[G]$  is a first-order formula with occurrences of a subformula  $G$  in which  $x$  does not occur free and whose free variables are not in scope of a quantifier within  $F[G]$ . Formulas  $F[\top]$  and  $F[\perp]$  denote  $F[G]$  with all the occurrences of  $G$  replaced by  $\top$  or  $\perp$ , respectively. Then*

- (i)  $\exists x F[G] \equiv (G \vee \exists x F[\perp]) \wedge (\neg G \vee \exists x F[\top])$ .
- (ii)  $\forall x F[G] \equiv (G \wedge \forall x F[\top]) \vee (\neg G \wedge \forall x F[\perp])$ .

Obviously, applying the expansions according to Prop. 20 could be immediately followed by truth-value simplification (EQ 9–EQ 16), which is actually assumed in the original presentation by Quine. The version of Prop. 20.ii given in [Qui45], is actually a generalization of the well-known propositional Shannon expansion.

<sup>18</sup> *Antiprenex form* has been used in [Bib74] for the first variant.

The method of [Qui45] proceeds by exhaustively rewriting innermost subformulas that match the precondition of the expansion.

## 19 Other Methods for Deciding Relational Monadic Formulas

Alternative decision methods for MON formulas include resolution: Equipped with an appropriate ordering and condensation, it decides MON formulas, although the associated Herbrand universe might be infinite due to Skolemization [Fer+01]. A superposition-based decision method for  $\text{MON}_=$  is given in [BGW93].

Deciding satisfiability is for MON and for  $\text{MON}_=$  NEXPTIME-complete, as presented in [BGG97, Sect. 6.2] along with more fine-grained results. Upper bounds makes use of the fact that a satisfiable  $\text{MON}_=$  (MON, resp.) formula has a model whose cardinality is at most  $q2^m$  ( $2^m$ , resp.), where  $q$  is the quantifier rank and  $m$  is the number of predicates. The underlying method of [Lew80] for deciding MON verifies a given interpretation by repeatedly constructing an inner form with respect to a single innermost quantifier occurrence and then replacing the corresponding obtained quantified subformulas with  $\top$  or  $\perp$  according to the interpretation. The processing of an existential (universal, resp.) innermost quantifier proceeds by conversion of its argument formula to disjunctive (conjunctive, resp.) normal form, followed by distributing the quantifier over the conjunctive clauses (clauses, resp.) and then, in each conjunctive clause (clause, resp.), narrowing the quantifier scope to those literals that contain the quantified variable. Determination of the upper bound makes use of the fact that all atoms occurring in the intermediate formulas are already present in the input formula.

## 20 Normal Form with Respect to a Predicate

Behmann notes that for practical application it is not necessary to construct the *Eliminationshauptform* ((2) on p. 21), via the *fully developed* disjunctive normal form (as done in the methods described in the proofs of Theorem 2 and Lemma 14 and their correspondents for the case without equality), but that it suffices if just the predicate to eliminate is separated from other predicates with respect to the associated variables [Beh22a, p. 201]. Although Behmann remarks that the corresponding forms, developed just with respect to a given predicate, observe a number of strange laws and are of not much less theoretical and practical importance than normal forms, he does not pursue this issue in [Beh22a],<sup>19</sup> but just gives an example:

$$\exists p \exists x (qx \vee rx) \wedge px \tag{11}$$

immediately matches the *Eliminationshauptform* with  $a = b = d = 0$ ,  $c = 1$  and  $C_1[x] = qx \vee rx$  such there is no point in distributing the conjunction with  $px$  over  $qx \vee rx$ .

<sup>19</sup> So far, there has been no publication or manuscript by Behmann identified where he would present the indicated material.

There seem at least superficial relationships with predicate elimination techniques that involve concepts like *standardized* and *in good scope* which are parameterized with a predicate to eliminate [Con06] or the conversion to formulas where conjuncts are *linkless outside a set of predicates*, a property that permits to distribute existential quantification upon predicates not in the set (“outside” the set) over conjunction [Wer09].

## 21 An Application of Elimination: Modeling Syllogistic Reasoning

As noted in [Cra08], the view of syllogisms as instances of elimination problems where the conclusion is the result of eliminating the middle term from the conjunction of the premises is due to Boole [Boo54]. Also Schröder pursues this representation of syllogisms [Sch91, § 42–44]. If the three terms involved in syllogisms are expressed as unary predicates, the premises and conclusions correspond to sentences of first-order logic in the relational monadic fragment. Behmann [Beh22a, § 17] exemplarily demonstrates his second-order quantifier elimination method with modeling some syllogisms and related statements by Schröder. Here, these examples are shown with some intermediate steps that indicate the involved rewritings.<sup>20</sup>

Syllogism *Ferio*.

$$\begin{aligned} & \exists q (\forall x \neg qx \vee \neg px) \wedge (\exists x rx \wedge qx) \\ \equiv & \exists u ru \wedge \exists q (\forall x \neg qx \vee \neg px) \wedge (\forall x x \neq u \vee qx) \\ \equiv & \exists u ru \wedge \neg pu. \end{aligned} \quad (12)$$

Syllogism *Darapti*. Here the implicitly understood non-emptiness of  $q$  is added as a third auxiliary premise  $\exists x qx$ .

$$\begin{aligned} & \exists q (\forall x \neg qx \vee px) \wedge (\forall x \neg qx \vee rx) \wedge (\exists x qx) \\ \equiv & \exists u \exists q (\forall x \neg qx \vee px) \wedge (\forall x \neg qx \vee rx) \wedge (\forall x x \neq u \vee qx) \\ \equiv & \exists u pu \wedge ru. \end{aligned} \quad (13)$$

Behmann shows the following example from [Sch91, p. 361], with the premises “Some  $p$  are not  $q$ . Some  $q$  are not  $r$ .” Since the first conjuncts of both premises are particular and negative no conclusions would be expected in the sense of traditional syllogistic reasoning. Elimination, however, allows to conclude that there exist two distinct individuals, one in  $p$  and the other not in  $r$ :

$$\begin{aligned} & \exists q (\exists x px \wedge \neg qx) \wedge (\exists x qx \wedge \neg rx) \\ \equiv & \exists u \exists v pu \wedge \neg rv \wedge \exists q (\forall x x \neq u \vee \neg qx) \wedge (\forall x qx \vee x \neq v) \\ \equiv & \exists u \exists v pu \wedge \neg rv \wedge u \neq v. \end{aligned} \quad (14)$$

<sup>20</sup> In addition, in [Beh22a] a “singular” reading of syllogisms is discussed, where the middle term is understood as individual instead of a predicate and elimination is not involved in modeling.

The following example given by Behmann is a “composed syllogism” from Schröder [Sch91, p. 283], which involves elimination of two predicates  $q_1$  and  $q_2$ .

$$\begin{aligned}
& \exists q_1 \exists q_2 (\forall x \neg q_1 x \vee px) \wedge (\forall x \neg q_2 x \vee rx) \wedge (\exists x q_1 x \wedge q_2 x) \\
\equiv & \exists q_1 (\forall x \neg q_1 x \vee px) \wedge \exists q_2 \wedge (\forall x \neg q_2 x \vee rx) \wedge (\exists x q_1 x \wedge q_2 x) \\
\equiv & \exists q_1 (\forall x \neg q_1 x \vee px) \wedge \exists u q_1 u \wedge \\
& \quad \exists q_2 \wedge (\forall x \neg q_2 x \vee rx) \wedge (\forall x x \neq u \vee q_2 x) \\
\equiv & \exists q_1 (\forall x \neg q_1 x \vee px) \wedge (\exists u q_1 u \wedge ru) \\
\equiv & \exists u ru \wedge \exists q_1 (\forall x \neg q_1 x \vee px) \wedge (\forall x x \neq u \vee q_1 x) \\
\equiv & \exists u ru \wedge pu.
\end{aligned} \tag{15}$$

## 22 Polyadic Formulas Allowing Monadic Elimination Techniques and Handling of Auxiliary Definitions

The elimination properties of monadic logic also apply in certain cases where involved predicates have more than one argument, for example, if arguments except of one are instantiated with a constant or with a variable that is free in the scope of the respective predicate quantifiers. Behmann gives in [Beh22a, § 19] several examples for applying his elimination method to such formulas. His first example is from [Sch95, p. 491]:

$$\begin{aligned}
& \exists p (\forall z fxz \vee pzy \vee hzy) \wedge (\forall z gxz \vee \neg pzy \vee hzy) \\
\equiv & \exists p' (\forall z f'z \vee p'z \vee h'z) \wedge (\forall z g'z \vee \neg p'z \vee h'z) \\
\equiv & \forall z f'z \vee g'z \vee h'z \\
\equiv & \forall z fxz \vee gxz \vee hzy,
\end{aligned} \tag{16}$$

where the following shorthands are used to hide the free variables from view:  $f'z = fxz$ ,  $g'z = gxz$ ,  $h'z = hzy$ ,  $p'z = pzy$ . The next example is from [Sch95, p. 308]:

$$\begin{aligned}
& \forall p (\exists u pxu \wedge fuy) \vee (\forall v \neg pxv \vee gvy) \\
\equiv & \neg \exists p' (\forall u \neg p'u \vee \neg f'u) \wedge (\exists v p'v \wedge \neg g'v) \\
\equiv & \neg \exists v \neg g'v \wedge \exists p' (\forall u \neg p'u \vee \neg f'u) \wedge (\forall u u \neq v \vee p'u) \\
\equiv & \neg \exists v \neg g'v \wedge \neg f'v \\
\equiv & \forall v gvy \vee fvy,
\end{aligned} \tag{17}$$

where the following shorthands are used (for arbitrary variables  $z$ ):  $f'z = fzy$ ,  $g'z = gzy$ ,  $p'z = pxz$ . The following formula from [Sch95, p. 510] is shown as an example where elimination with the method of [Beh22a] fails, since a proper relation between  $u$  and  $v$  is involved.

$$\forall p pxv \vee (\exists v (\forall u \neg pxu \vee fuv) \wedge gvy).$$

The final example given in [Beh22a, § 19] is from [Sch95, p. 545] and involves two instances of the binary predicate  $p$  to eliminate where the bound variable

occurs in different argument positions:

$$\begin{aligned}
& \exists p (\forall z f x z \vee p z y) \wedge (\forall z \neg p x z \vee g z y) \\
\equiv & \exists p' \exists q' (\forall z f' z \vee p' z) \wedge (\forall z \neg q' z \vee g' z) \wedge ((p' x \wedge q' y) \vee (\neg p' x \wedge \neg q' y)) \quad * \\
\equiv & (\exists p' \exists q' ((\forall z f' z \vee p' z) \wedge (\forall z \neg q' z \vee g' z) \wedge p' x \wedge q' y)) \vee \\
& (\exists p' \exists q' ((\forall z f' z \vee p' z) \wedge (\forall z \neg q' z \vee g' z) \wedge \neg p' x \wedge \neg q' y)) \\
\equiv & ((\exists p' (\forall z f' z \vee p' z) \wedge p' x) \wedge (\exists q' (\forall z \neg q' z \vee g' z) \wedge q' y)) \vee \\
& ((\exists p' (\forall z f' z \vee p' z) \wedge \neg p' x) \wedge (\exists q' (\forall z \neg q' z \vee g' z) \wedge \neg q' y)) \\
\equiv & (\exists q' (\forall z \neg q' z \vee g' z) \wedge q' y) \vee \quad ** \\
& (\exists p' (\forall z f' z \vee p' z) \wedge \neg p' x) \\
\equiv & (\exists q' (\forall z \neg q' z \vee g' z) \wedge (\forall z z \neq y \vee q' z)) \vee \\
& (\exists p' (\forall z f' z \vee p' z) \wedge (\forall z z \neq x \vee \neg p' z)) \\
\equiv & (\forall z z \neq y \vee g' z) \vee (\forall z f' z \vee z \neq x) \\
\equiv & g' y \vee f' x \\
\equiv & g y y \vee f x x,
\end{aligned} \tag{18}$$

where the following shorthands are involved:  $f'z = f x z$ ,  $g'z = g z y$ ,  $p'z = p z y$ ,  $q'z = p x z$ . The two instances of  $p$  are represented by different shorthands  $p'$  and  $q'$  that are related by the equivalence  $p'x \leftrightarrow q'y$ , which is added (expressed with disjunction and conjunction) in step (\*). Step (\*\*) is obtained by deleting two subformulas for which elimination yields  $\top$ .

So far, the contraction and expansion with the shorthand predicates has been handled on the meta level. Second-order quantification would allow to understand the introduction and elimination of such definitions as equivalence preserving transformations, with Ackermann's Lemma [Ack35a], a foundation of modern elimination methods such as [DLS97; CGV06; Sch12; KS13], as a special case. The basis for this understanding of introducing and expanding of auxiliary definitions is the following property, which easily follows from Lemma 3:

**Proposition 21 (Eliminability of Expandable Definitions).** *Let  $p$  be a unary predicate and let  $F$  be a first-order formula in which  $p$  does not occur. It then holds that*

$$\exists p \forall x p x \leftrightarrow F \equiv \top.$$

With Prop. 21, the following proposition can be derived, which explicates the handling of auxiliary definitions by means of equivalence preserving formula transformations:

**Proposition 22 (Introduction and Elimination of Definitions).** *Let  $p$  be a unary predicate, let  $x$  be an variable and let  $G[x]$  be a first-order formula in which  $p$  does not occur. For a constant or variable  $t$ , let  $G[t]$  denote  $G[x]$  with all free occurrences of  $x$  replaced by  $t$ . Let  $F[G[t_1], \dots, G[t_n]]$  be a first-order formula in which  $p$  does not occur and which has  $n$  occurrences of subformulas, instantiated with  $G[t_1], \dots, G[t_n]$ , respectively, neither of them in a context where a variable that occurs free in  $G[x]$  is bound. Let  $F[pt_1, \dots, pt_n]$  denote the same formula with the indicated occurrences  $G[t_i]$  replaced by  $pt_i$ . Then*

$$F[G[t_1], \dots, G[t_n]] \equiv \exists p (\forall x p x \leftrightarrow G[x]) \wedge F[pt_1, \dots, pt_n].$$

Prop. 22 can be applied from left to right to introduce auxiliary predicates  $p$  and from right to left to expand them, by replacing *all* occurrences of  $p$  with their definienda and then dropping the definition.

In [Beh22a, § 19], the shorthand predicates are not explicitly handled in this way, although the notation, where they are specified with the equivalence symbol  $\leftrightarrow$ , might suggest that this could be the underlying intuition. In later works he explicitly applies a variant of Prop. 22 to introduce definitions of nullary predicates, presented in the manuscript [Manuscript M12] as a second-order adaption of EQ 32 and EQ 33 (see also (35) in Sect. 27).

In the case where  $p$  occurs in  $F[pt_1, \dots, pt_n]$  just with, say, positive polarity, it holds that

$$\begin{aligned} & \exists p (\forall x px \leftrightarrow G[x]) \wedge F[pt_1, \dots, pt_n] \\ \equiv & \exists p (\forall x px \rightarrow G[x]) \wedge F[pt_1, \dots, pt_n], \end{aligned} \quad (19)$$

which leads to Ackermann's Lemma [Ack35a], shown here for unary predicates  $p$ :

**Proposition 23 (Ackermann's Lemma).** *Assume the setting of Prop. 22 and that all the indicated subformula occurrences in  $F[G[t_1], \dots, G[t_n]]$  (or, equivalently, in  $F[pt_1, \dots, pt_n]$ ) have the same polarity  $P$ . Then*

$$\begin{aligned} \exists p (\forall x px \rightarrow G[x]) \wedge F[pt_1, \dots, pt_n] & \equiv F[G[t_1], \dots, G[t_n]], \text{ if } P \text{ is positive.} \\ \exists p (\forall x px \leftarrow G[x]) \wedge F[pt_1, \dots, pt_n] & \equiv F[G[t_1], \dots, G[t_n]], \text{ if } P \text{ is negative.} \end{aligned}$$

The Basic Elimination Lemma (Lemma 3) is obviously an instance of Ackermann's Lemma. Vice versa, Ackermann's Lemma can be proven such that the only elimination step is performed according to the Basic Elimination Lemma, or according to Prop. 21.

## 23 Ackermann's Quantifier Switching

In [Ack35b], a short sequel to [Ack35a], Ackermann shows a precondition which allows to move existential predicate quantification to the right of universal individual quantification, where the arity of the quantified predicate is reduced:

**Lemma 24 (Ackermann's Quantifier Switching).** *Let  $p$  be a predicate with arity  $n+1$ , where  $n \geq 0$ . Let  $F = F[pxt_{11} \dots t_{1n} \dots pxt_{m1} \dots t_{mn}]$ , where  $m \geq 1$ , be a formula of second-order logic in which  $p$  has the exactly  $m$  indicated occurrences. Assume further that  $p$  and  $x$  occur only free in  $F$ . Let  $q$  be a predicate with arity  $n$  that does not occur in  $F$  and let  $F[qt_{11} \dots t_{1n}, \dots, qt_{m1} \dots t_{mn}]$  denote  $F$  with each occurrence  $pxt_{ij} \dots t_{ij}$  of  $p$  replaced by  $qt_{ij} \dots t_{ij}$ , for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ . Under the assumption of the axiom of choice it then holds that*

$$\begin{aligned} & \exists p \forall x F[pxt_{11} \dots t_{1n}, \dots, pxt_{m1} \dots t_{mn}] \\ \equiv & \forall x \exists q F[qt_{11} \dots t_{1n}, \dots, qt_{m1} \dots t_{mn}]. \end{aligned}$$

Van Benthem [Ben83, p. 211] mentions this equivalence with application from right to left to achieve prenex form with respect to second-order quantifiers.

Ackermann applies this equivalence in [Ack35b] to avoid Skolemization and to convert formulas such that monadic techniques or Ackermann’s Lemma become applicable. He shows five examples from [Sch05]. The first of these rewrites the input formula such that elimination methods for MON becomes applicable. The formula to which these are applied contains binary predicates, but as for Behmann’s examples in Sect. 22, in all occurrences of binary predicates one of the arguments is free in the scope of the predicate quantifier to eliminate. The example proceeds in the following steps:

$$\begin{aligned}
& \exists f (\forall x \exists y \, axy \wedge fxy) \wedge (\forall x \exists z \, bxz \wedge \neg fzx) \\
\equiv & \exists f \forall x (\exists y \, axy \wedge fxy) \wedge (\exists z \, bxz \wedge \neg fzx) && EQ \ 21 \\
\equiv & \forall x \exists g (\exists y \, axy \wedge gy) \wedge (\exists z \, bxz \wedge \neg gz) && Lemma \ 24 \\
\equiv & \forall x \exists y \exists z \, axy \wedge bxz \wedge y \neq z. && Lemma \ 19
\end{aligned} \tag{20}$$

Ackermann notes in [Ack35b] that he has discussed this example already in [Ack35a] to illustrate another method: Applying the variant of resolution-based elimination from [Ack35a] (of which the modern SCAN [GO92] is a refinement – see [NOS99]), which involves conversion to a universal formula by means of Skolem functions and un-Skolemization after elimination.

Indeed, Ackermann motivates his quantifier switching technique by the fact that the introduction of Skolem functions (*Belegungsfunktionen*) leads to such intricate tasks that one would like to avoid these functions for those special cases where the results can also be obtained in other ways. In this sense, he also writes to Behmann in October 1934 that he does no longer consider Skolem functions as an advantage.<sup>21</sup>

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<sup>21</sup> *Im übrigen halte ich neuerdings die Einführung der Belegungsfunktionen für keinen Vorteil mehr; die mit Hilfe der Belegungsfunktionen auszudrückenden Probleme, so einfach sie sich auch symbolisch ausdrücken lassen, werden so schwierig, dass ich da kein Weiterkommen sehe. Andererseits lassen sich in speziellen Fällen, wie bei meinem Beispiel (26), erzielten Resultate ebensogut durch Anwendung der Formel*

$$\forall x \exists y \, gxy \equiv \exists f (\forall x \exists y \, fxy) \wedge (\forall x \forall y \, \neg fxy \vee gxy)$$

*und entsprechender Formeln für mehr Variable gewinnen.* Letter from Wilhelm Ackermann to Heinrich Behmann, 29 October 1934 [Letter L7]. Formulas are rendered in modern notation. The mentioned *Beispiel (26)* is the example reproduced above as (20). The shown equivalence follows from Lemma 24 and 19, but it does not immediately match with the referenced example.



Part IV

Towards Elimination for  
Relations: The  
Correspondence between  
Behmann and Ackermann  
1927–1934



## 24 Introduction to Part IV

In this part Behmann's follow-up works to [Beh22a], his related correspondence with Ackermann and his presentation of the resolution-based elimination method by Ackermann [Ack35a] is summarized. The topic is the decision problem for relations (*Entscheidungsproblem für Beziehungen*), that is, the decision problem for formulas which contain predicates of arity two or more. The approach is, as for the monadic case in [Beh22a], to apply second-order quantifier elimination techniques.

In [Beh22a, p. 226f], Behmann conjectures that for the extension of the decision problem to arbitrary relations and higher concepts it is questionable whether the elimination problem can serve further as a suitable basis, justified on the following considerations: "If two classes  $\alpha$  and  $\beta$  satisfy a condition that can be specified purely logical and involves some variable classes – for example, that there is a third class which contains  $\alpha$  as subclass and is itself contained in  $\beta$  as subclass –, then we know indeed that we can express such a condition certainly also without mentioning such variable classes. The matter is, however, as it seems, much more intricate if a variable *relation*<sup>22</sup> is allowed, as, for example, at the statement that two classes  $\alpha$  and  $\beta$  have the same cardinality, that is, that by a certain relation the elements of one class can be mapped in a one-to-one correspondence to that of the other one. Here one does not see a possibility to express the condition that two classes have the same cardinality in general and without reference to such a variable relation. Presumably, again a completely new idea is required here."<sup>23</sup>

Behmann gave in September 1926 at the *Jahresversammlung der Deutschen Mathematiker-Vereinigung* a talk on the decision problem and the logic of relations (*Entscheidungsproblem und Logik der Beziehungen*). Its abstract, published as [Beh27a], aroused the curiosity of Ackermann, who wrote in August 1928 to

<sup>22</sup> As pointed out later by Ackermann [Ack35a, p. 393], it is, however, not essential that the predicate *to eliminate* has an arity larger than one – the same difficulties arise if the predicate to eliminate is unary but other predicate have arities with two and more arguments.

<sup>23</sup> *Was die Erweiterung des Entscheidungsproblems auf beliebige Beziehungen und höhere Begriffe angeht, so erscheint es immerhin fraglich, ob auch hier das Eliminationsproblem weiterhin als geeignete Grundlage dienen können wird, und zwar auf Grund der folgenden Überlegung: Genügen etwa zwei Klassen  $\alpha$  und  $\beta$  einer rein logisch angebbaren Bedingung, innerhalb deren irgendwelche veränderliche Klassen vorkommen – sagen wir z. B. derjenigen, daß es eine dritte Klasse gibt, die  $\alpha$  als Teilklasse enthält und ihrerseits als Teilklasse in  $\beta$  enthalten ist –, so wissen wir allerdings, daß wir eine solche Bedingung gewiß auch ohne Erwähnung derartiger veränderlicher Klassen auszudrücken vermögen. Die Sache liegt indessen, wie es scheint, wesentlich verwickelter, sobald eine veränderliche Beziehung in Frage kommt, wie z. B. bei der Aussage, daß die Klassen  $\alpha$  und  $\beta$  gleichzählig sind, d. h. daß durch eine gewisse Beziehung die Elemente der einen denen der anderen umkehrbar eindeutig zugeordnet werden. Hier sieht man durchaus keine Möglichkeit, die Bedingung der Gleichzähligkeit zweier Klassen allgemein ohne einen Hinweis auf eine derartige veränderliche Beziehung auszudrücken. Vermutlich wird es hier also wiederum eines ganz neuen Gedankens bedürfen. [Beh22a, p. 226f].*

Behmann, initiating a correspondence that lasted to November 1928 and comprises five letters. Related topics were discussed later in two letters, one sent by Behmann upon receiving the offprint of [Ack35a], and Ackermann’s reply. Their correspondence, as far as archived in [BehNL], then only continues in January 1953, with five more letters until December 1955, where different topics are discussed (a complete register of their correspondence in [BehNL] is provided in Sect. 31).

Further sources of this presentation include a manuscript [Manuscript M11] for the abstract [Beh27a] and an unpublished manuscript from December 1934 [Manuscript M12] *Ein wichtiger Fortschritt im Entscheidungsproblem der Mathematischen Logik* (*An Important Progress in the Decision Problem of Mathematical Logic* with subtitle (*Ackermann Math. Annalen 110 S.390*), referring to [Ack35a].

The addressed topics include the use of Skolemization as well as the early form of resolution by Ackermann [Ack35a], applied to express results of second-order quantifier elimination on universal formulas by a possibly infinite set of formulas. Resolution-based second-order quantifier elimination has been considered in modern times with the SCAN algorithm [GO92]. The use of logics extended with a fixpoint operator to express possibly infinite elimination results has been investigated in [NS98] and is today one of the core techniques for second-order quantifier elimination (or “forgetting”) in description logics [KS13]. So far, however, the fixpoint approach is based not on Ackermann’s resolution method, but on Ackermann’s Lemma, another result from [Ack35a] (see Prop. 23 on p. 53). A further work that explicitly relates to Ackermann’s resolution-based method is [Cra60]. In his manuscript from 1934 and his letter to Ackermann, Behmann suggests a graph representation for the possibly infinite set of resolvents.

We use the same modern syntax for the presentation as in the other parts, but keep the Greek letters  $\varphi, \chi, \psi$ , and  $\alpha, \beta, \gamma$  from the original documents. Ackermann originally writes predicates in upper case, which are rendered here in lower case. Detailed information about the consulted manuscripts and letters, as well as a summary of other topics discussed in the correspondence is provided in Sect. 30 and Sect. 31.

## 25 First Considerations on Elimination for Universal Formulas

In the abstract [Beh27a] of his talk *Entscheidungsproblem und Logik der Beziehungen* (*Decision Problem and Logic of Relations*) given on 23 September 1926 at the *Jahresversammlung der Deutschen Mathematiker-Vereinigung* in Düsseldorf, Behmann considers elimination of an existential second-order quantifier upon a predicate  $\varphi$  with arity two or larger applied to a universal first-order formula, that is, elimination in

$$\exists\varphi\forall x_1, \dots, \forall x_n F, \tag{21}$$

where  $\varphi$  is a predicate with arity  $\geq 2$  and  $F$  is a first-order formula. Behmann states in [Beh27a], that this problem allows reduction to the following sequence of problems:

$$\begin{aligned} & \exists\varphi\forall x F[\varphi x, x], \\ & \exists\varphi\exists\chi\forall x\forall y F[\varphi x, \chi y, x, y], \\ & \exists\varphi\exists\chi\exists\psi\forall x\forall y\forall z F[\varphi x, \chi y, \psi z, x, y, z], \\ & \dots^{24} \end{aligned} \tag{22}$$

where the quantified predicates are unary and the formulas  $F[\dots]$  are first-order and such that all occurrences of the quantified predicates have the indicated variable as argument. In addition, the individual variables themselves are listed in the square brackets, indicating that they might also have further occurrences in the formula.<sup>25</sup> Behmann further claims in [Beh27a] that formulas of that form would allow elimination of the predicate quantifiers.

The solution for the first component of the sequence is

$$\forall x\exists p F[p, x], \tag{23}$$

or

$$\forall x F[\perp, x] \vee F[\top, x], \tag{24}$$

respectively, where  $p$  is a fresh nullary predicate and  $F[G]$  denotes  $F[\varphi x]$  with all occurrences of  $\varphi x$  replaced by  $G$ . The equivalence of the first component to (23) can indeed be obtained by Ackermann's arity reduction, Lemma 24, which seems implicitly applied here by Behmann.

In the manuscript [Manuscript M11] for [Beh27a], the following formula is shown as intended solution for the second component:

$$(\forall x\exists p\forall y\exists q F[p, q, x, y]) \wedge (\forall y\exists q\forall x\exists p F[p, q, x, y]). \tag{25}$$

For the second component, Behmann's claim is false. On 16 August 1928 Ackermann writes to him [Letter L1], requesting clarification by giving the following example:

$$\begin{aligned} & \exists\varphi\forall x\forall y (\varphi x \vee \neg\varphi y \vee \alpha xy) \wedge \\ & \quad (\varphi x \vee \beta x) \quad \wedge \\ & \quad (\neg\varphi x \vee \gamma x), \end{aligned} \tag{26}$$

which can be written as instance of the second component of (22):

$$\begin{aligned} & \exists\varphi\exists\chi\forall x\forall y (\varphi x \vee \neg\chi y \vee \alpha xy) \quad \wedge \\ & \quad (\varphi x \vee \beta x) \quad \wedge \\ & \quad (\chi y \vee \gamma y) \quad \wedge \\ & \quad (\varphi x \vee \chi y \vee x \neq y) \quad \wedge \\ & \quad (\neg\varphi x \vee \neg\chi y \vee x \neq y). \end{aligned} \tag{27}$$

<sup>24</sup> In a letter to Alonzo Church, dated 15 April 1937, Behmann refers in a broader context to this sequence of formulas. See Sect. 32.4.

<sup>25</sup> [Beh27a] contains some obvious printing errors that have been quietly corrected here – see discussion of manuscript [Manuscript M11] in Sect. 30.

Behmann replies on 21 August 1928 [Letter L2] from the holiday island Föhr at the German North Sea coast that some years ago he had achieved a certain point in his investigations of the decision problem for relations but had to leave it unattended since then. He announces to send his results to Ackermann as soon as he returns to Halle. Behmann continues that he became aware only between the talk and the correction that his claim to have settled the case of elimination for universal individual quantifiers was false. He had informed Bieberbach<sup>26</sup> but decided to keep the statement for historic accuracy in essence in the abstract, with the intention of addressing the issue later in a publication. Behmann notes that after many void attempts to find a satisfying proof, he came to a specific example where his conjectured solution was in fact weaker, that is, implied but not equivalent to the given second-order formula. He remarks that Ackermann's example would be of great interest, since it might show that the schema of progressive concept elimination would not be sufficient and a fundamentally different way has to be searched. Ackermann gave later in [Ack35a, Section 3] a proof that the considered elimination problems on formulas with only universal individual quantifiers can not be solved in general.

On 1 September 1928 [Letter L3] Ackermann thanks Behmann for his informative reply. He notes that it took a load from his mind since he had already struggled a lot with the elimination problem in the case where all individual quantifiers are universal.<sup>27</sup> He continues that the probability that an elimination is not always possible had emerged for him at first from the following example:

$$\exists f \forall x \forall y \, axy \wedge ((fx \vee fy) \wedge (\neg fx \vee \neg fy)). \quad (28)$$

The following formulas are then obtained as consequences:

$$\begin{aligned} & \forall x \, axx, \\ & \forall x \forall y \forall z \, axy \vee ayz \vee axz, \\ & \forall x \forall y \forall z \forall u \forall v \, axy \vee ayz \vee a(z, u) \vee auv \vee axv, \\ & \dots \end{aligned} \quad (29)$$

but one searches in vain for an expression that contains all of these as consequences. Ackermann writes that does not yet have an exact proof that the elimination can not be performed, but hopes to find one within some weeks, even if he expects it to be quite complicated.

The example (28) is also used later by Ackermann, as Example (20) on p. 410 in [Ack35a], to illustrate his resolution-based elimination method and discussed in his letter to Behmann from October 1934 [Letter L7].

<sup>26</sup> Ludwig Bieberbach (1886–1982) was the editor of the respective number of the *Jahresbericht der Deutschen Mathematiker-Vereinigung*.

<sup>27</sup> *Mir ist dadurch ein Stein vom Herzen gefallen, da ich mich mit dem Eliminationsproblem in dem Falle, daß alle Dingoperatoren allgemein sind, schon viel herumgequält hatte.* [Letter L3].

## 26 Involvement of Skolemization and Un-Skolemization

In the abstract [Beh27a] of the Düsseldorf talk Behmann goes beyond the case of universal individual quantifiers to cases where universal and existential quantifiers alternate, as for example in formulas of the form

$$\exists\varphi\forall x\exists y F[\varphi x, \varphi y, x, y]. \quad (30)$$

The problem would be solved, if the order of  $\forall x$  and  $\exists y$  could be switched, which would then allow to move  $\exists y$  further left, in front of the predicate quantifier  $\exists\varphi$ . Behmann remarks that this can be achieved indeed by Skolemization, which is introduced in [Beh27a] as a trick (*Kunstgriff*) described by Schröder in [Sch95, p. 512ff].<sup>28</sup> Behmann writes that instead of predicate quantifiers, now quantifiers upon the Skolem functions (*Belegungsoperatoren*) do appear, which can not be undone in general – at least with known means (what today is called un-Skolemization). He mentions that a peculiar extension of his representation schema succeeds in undoing Skolemization, essentially by allowing aside of the so far solely known and applied succession of quantifiers  $\forall$  and  $\exists$  more entangled linkages of them, associated with specific meanings. This seems to anticipate that un-Skolemization after predicate elimination can in some cases only obtained with Henkin quantifiers [GO92]. In the manuscript [Manuscript M11], he adds that these linkages concern in particular cycles, which, for example, do no longer comply with the transitivity of sooner or later. He concludes the talk abstract with the remark that control of the decision problem for the considered class of formulas then appears as equivalent to control of this extended representation schema.

After reading the elaborate transcript of the Düsseldorf talk sent to him by Behmann,<sup>29</sup> Ackermann expresses in his letter dated 1 November 1928 [Letter L5] doubts about Behmann’s use of Skolemization and un-Skolemization, conjecturing that it amounts to a re-expression of the decision problem with functions, which can be used to encode predicates: That  $fx$  holds can be expressed as  $\varphi x = 0$ , where  $\varphi$  is a function with range  $\{0, 1\}$  associated with  $f$ .<sup>30</sup>

<sup>28</sup> Second-order Skolemization allows to convert an existential individual quantifier in the scope of universal quantifiers to an existential function quantifier that is left of the universal quantifiers. The underlying equivalence is

$$\forall x_1 \dots x_n \exists y F[y] \equiv \exists f \forall x_1 \dots x_n F[f x_1 \dots x_n],$$

where  $f$  is a  $n$ -place function symbol that does not occur in  $F[y]$  and  $F[f x_1 \dots x_n]$  is obtained from  $F[y]$  by replacing all free occurrences of  $y$  with  $f x_1 \dots x_n$ .

<sup>29</sup> So far, that transcript could not be located – see remarks on [Letter L4] in Sect. 31 and on [Manuscript M10] in Sect. 30. In 1934 Behmann also gave a detailed account of his earlier work on the decision problem for relations in letter [Letter L6] and manuscript [Manuscript M12], which is discussed below in Sect. 27.

<sup>30</sup> *Glauben Sie aber nicht, daß das Problem, das sich als Schlußproblem Ihrer Untersuchung darstellt, ebenso schwierig ist wie das allgemeine Entscheidungsproblem? Die kompliziertesten Verkettungen von Operatoren, die dabei auftreten, kommen doch*

In that letter, Ackermann also reports that he got notice that Löwenheim in Berlin has exact proofs that elimination can not be performed on certain logic expressions, and that it would be desirable that Löwenheim would publish these because it would demonstrate that the existing way does not lead further on. After remarking that the investigation of a more modest question seems to him most promising, Ackermann sketches an idea for a method to verify valid universal formulas (see discussion of [Letter L5] in Sect. 31).

## 27 Behmann’s Representation of Ackermann’s Resolution-Based Method

Today, the two prevailing approaches to second-order quantifier elimination are the so-called direct methods, based on Ackermann’s Lemma [DLS97], and the resolution-based approach of the SCAN algorithm [GO92]. Both can be traced back to [Ack35a], where the lemma underlying the direct approach is defined and a variant of the – seemingly rediscovered [NOS99] – resolution-based approach is elaborated. Further works related to Ackermann’s resolution-based method have been already mentioned in Sect. 24.

Aside of the correspondence with Behmann, a published selection from the correspondence of Wilhelm Ackermann edited by his son Hans-Richard Ackermann [Ack83] gives further hints on the “pre-history” of Ackermann’s important paper [Ack35a]: He sent the manuscript in 1933 to Bernays (Letter from Bernays to Ackermann, 24 December 1933), who then recommended it to Hilbert for publication in *Mathematische Annalen*. Bernays sent six large pages with remarks about the manuscript, regarding content as well as presentation, to Ackermann, which he considered for the version submitted to Blumenthal<sup>31</sup> (Letter from Ackermann to Bernays, 14 January 1934). Accordingly, the submission date given in the publication is 13 January 1934.

After receiving the offprint of Ackermann’s paper, Behmann writes on 22 October 1934 to Ackermann [Letter L6], congratulating him to his success, noting that the work shows to him that the proper access was quite hidden. Behmann mentions that he perceived the existing research line, which aimed at a generalization of the method of substitution (*Verallgemeinerung des Einsetzungsverfahrens*) as unsatisfying. He admires Ackermann’s methodical-technical generality by posing only little requirements on normalization (*Normierung*) of the given problem, however, as Behmann remarks, this entails the disadvantage that it

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*schließlich darauf hinaus, daß man statt der Prädikate und Relation Funktionen im Sinne der Mathematik einführt, d.h. eindeutige Zuordnungen von Individuen zu Individuen. Man kann dann aber gleich jedes Entscheidungsproblem in dieser Form geben, indem man ein Prädikat  $F(x)$  ersetzt durch eine derartige Funktion  $\varphi(x)$ , wo  $\varphi(x)$  entweder den Wert 0 oder 1 hat, und  $\varphi(x) = 0$  mit dem Bestehen von  $F(x)$  äquivalent ist. [Letter L5].*

<sup>31</sup> Otto Blumenthal (1876–1944) was the editor of the *Mathematische Annalen* responsible for Ackermann’s paper.



then seems not possible to clearly overview in general the totality of the resultants that are free from the predicate to eliminate (*die Gesamtheit der Zähl- ausdrücke II allgemein zu übersehen*) and to symbolize the resultant in a suitable transparent way. Behmann proceeds with a technical presentation which relates Ackermann's results to his own earlier work and suggests alternative more normalized representations of the elimination resultants obtained by Ackermann's resolution-based method.

In his reply of 29 October 1934 [Letter L7], Ackermann expresses his pleasure about Behmann's acknowledgment, in particular because Behmann's work from 1922 [Beh22a] was, at its time, the impetus for his investigation of the problem.<sup>32</sup> He then discusses the issue of getting a complete overview on the resultants (*vollkommene Übersicht über die Resultanten*), raised by Behmann.

Following the exchange with Ackermann, Behmann prepared a manuscript [Manuscript M12], titled *Ein wichtiger Fortschritt im Entscheidungsproblem der Mathematischen Logik* (*An Important Progress in the Decision Problem of Mathematical Logic*, with subtitle (*Ackermann Math. Annalen 110 S.390*), dated 14 December 1934, where the technical material in part overlaps with the letter [Letter L6]. Here he gives a more detailed presentation of his suggestions to represent the resultants obtained by Ackermann's method.

Behmann's manuscript [Manuscript M12] begins with a comprehensive introductory part. After specifying notation and introducing the decision problem, Behmann sketches two methods to decide propositional validity: the method of truth-tables (*Verfahren der Einsetzungsproben*) and the method of conversion to a conjunctive normal form (*Verfahren der konjunktiven Normalform*), which is valid if and only if each of its clauses contains a literal and its complement. He then summarizes his method for second-order quantifier elimination in relational monadic formulas with equality, attributed by him to *Löwenheim, Skolem, Behmann*. Quantified predicates are eliminated successively from the inside of the formula by equivalence preserving transformations by a system of deterministic computation rules (*durch äquivalente Umformung durch ein System zwangsläufiger Rechenvorschriften*). Behmann speaks there also of "logic algorithm" (*logischer Algorithmus*). The result expresses "the sentence to test is true (or false) for all domain cardinalities with exception (when appropriate) of the finite number of cardinalities  $m, n, \dots$ ". For sufficiently large finite and for all infinite domains the value of the sentence is the same.

A second approach to decide a sentence is then sketched: It can be tested for particular cardinalities of the individual domain, based on the fact that there are theorems that allow to determine a suitable upper bound from its symbolic structure. The sentence is then repeatedly evaluated for each domain cardinality,

<sup>32</sup> *Für Ihren Brief und die freundlichen Worte über meine Arbeit meinen besten Dank. Ich freue mich über die Anerkennung um so mehr, als Ihre Arbeit in den Math. Ann. 86 seiner Zeit mir den Anstoss gegeben hat, mich mit dem Problem näher zu beschäftigen* [Letter L7]. In 1922 Ackermann finished his basic studies (*Grundstudium*) at Göttingen [Göt22; Ack83].

up to the previously determined bound. Behmann gives a short example, but concludes that this method will never be applied in practice.

Behmann now turns to adding predicates with arities larger than one, in his words, *variable relations between individuals* (*variable Beziehungen zwischen Individuen*). He discusses the possibility to generalize the method of successive elimination of quantified predicates, which lead to success in the monadic case. Duality justifies to consider just  $\exists\varphi$  as innermost predicate quantifier. Quantified predicates other than  $\varphi$  are then free in the argument formula and behave in the elimination process just like unquantified predicates. The formula considered for the elimination problem thus has the form

$$\exists\varphi F, \quad (31)$$

where in  $F$  is a first-order formula. The argument formula  $F$  can be assumed in prenex form. Behmann now restricts his considerations to the special case where in the quantifier prefix no existential quantifier is following a universal quantifier. Since the existential individual quantifiers can then be moved in front of the existential predicate quantifier, this restriction amounts to requiring the individual quantifier prefix of the argument of predicate quantification to be just universal. The considered elimination problem can then be brought into one of the following forms:

$$\begin{aligned} &\exists\varphi\forall x F[\varphi x, x], \\ &\exists\varphi\forall x\forall y F[\varphi x, \varphi y, x, y], \\ &\exists\varphi\forall x\forall y\forall z F[\varphi x, \varphi y, \varphi z, x, y, z], \\ &\dots \end{aligned} \quad (32)$$

where the  $F[\dots]$  are first-order and all occurrences of  $\varphi$  have the indicated variables as argument. In addition, the individual variables themselves are listed in the square brackets, indicating that they might also have further occurrences in the formula. (This series is like (22), p. 59, except that there a different predicate is used for each argument variable.) It suffices to consider predicate quantification just upon *unary* predicates, because quantification upon predicates with larger arity can be modeled by a new domain whose individuals are tuples of the original individuals. The solution for the first problem in the sequence (32) is

$$\forall x F[\perp, x] \vee F[\top, x], \quad (33)$$

where  $F[G, x]$  denotes  $F[\varphi x, x]$  with all occurrences of  $\varphi x$  replaced by  $G$ .

Behmann proceeds to discuss just the second problem in the sequence (32), because in his view there is no principal additional complication at the passage to the third and later problems. In analogy to EQ 32 and EQ 33 (*Umschreibungssatz*), the following second-order equivalences hold:

$$F[p] \equiv \forall q \neg(q \leftrightarrow p) \vee F[q] \equiv \exists q (q \leftrightarrow p) \wedge F[q]. \quad (34)$$

Behmann remarks that this holds for an *extensional* property of sentences ((*extensionale!*) *Eigenschaft von Aussagen*). Equivalence (34), given by Behmann, is

an instance of a version of Prop. 22 for nullary predicates. The equivalence actually applied by Behmann in the sequel can accordingly be expressed as follows, for formulas  $G$  under preconditions analogously to those in Prop. 22:

$$F[G] \equiv \forall p \neg(p \leftrightarrow G) \vee F[p] \equiv \exists p (p \leftrightarrow G) \wedge F[p]. \quad (35)$$

With (35), the following equivalences can be justified:

$$\begin{aligned} (1.) & \quad \exists \varphi \forall x \forall y F[\varphi x, \varphi y, x, y] \\ (2.) & \quad \equiv \exists \varphi \forall x \forall y \forall p \forall q \neg(p \leftrightarrow \varphi x) \vee \neg(q \leftrightarrow \varphi y) \vee F[p, q, x, y], \\ (3.) & \quad \equiv \exists \varphi \forall x \forall y (F[\perp, \perp, x, y] \vee \varphi x \vee \varphi y) \quad \wedge \\ & \quad (F[\perp, \top, x, y] \vee \varphi x \vee \neg \varphi y) \quad \wedge \\ & \quad (F[\top, \perp, x, y] \vee \neg \varphi x \vee \varphi y) \quad \wedge \\ & \quad (F[\top, \top, x, y] \vee \neg \varphi x \vee \neg \varphi y), \end{aligned} \quad (36)$$

where  $F[G, H, x, y]$  denotes  $F[\varphi x, \varphi y, x, y]$  with all occurrences of  $\varphi x$  replaced by  $G$  and all occurrences of  $\varphi y$  replaced by  $H$ . Step (3.) is obtained by expanding the Boolean quantifiers upon  $p$  and  $q$ , considering that  $\neg(\perp \leftrightarrow \varphi x) \equiv \varphi x$  and  $\neg(\top \leftrightarrow \varphi x) \equiv \neg \varphi x$ . Behmann now introduces the following form as shorthand for formulas of the form (3.) in (36):

$$\begin{aligned} \exists \varphi \forall x \forall y (fxy \vee \varphi x \vee \varphi y) & \quad \wedge \\ (gxy \vee \varphi x \vee \neg \varphi y) & \quad \wedge \\ (hxy \vee \neg \varphi x \vee \varphi y) & \quad \wedge \\ (kxy \vee \neg \varphi x \vee \neg \varphi y), & \end{aligned} \quad (37)$$

where  $f, g, h, k$  can be arbitrary binary relations on the underlying domain of individuals. In themselves,  $f, g, h, k$  are not subjected to any restrictions, however, certain constraints on them can be enforced. Formula (37) is equivalent to

$$\begin{aligned} \exists \varphi (\forall x \forall y (fxy \vee \varphi x \vee \varphi y) & \quad \wedge \\ (gxy \vee \varphi x \vee \neg \varphi y) & \quad \wedge \\ (hxy \vee \neg \varphi x \vee \varphi y) & \quad \wedge \\ (kxy \vee \neg \varphi x \vee \neg \varphi y)) & \quad \wedge \\ (\forall x \forall y (fyx \vee \varphi y \vee \varphi x) & \quad \wedge \\ (gyx \vee \varphi y \vee \neg \varphi x) & \quad \wedge \\ (hyx \vee \neg \varphi y \vee \varphi x) & \quad \wedge \\ (kyx \vee \neg \varphi y \vee \neg \varphi x)) & \quad \wedge \\ (\forall x \forall y x \neq y \vee \varphi x \vee \neg \varphi y) & \quad \wedge \\ (\forall x \forall y x \neq y \vee \neg \varphi x \vee \varphi y), & \end{aligned} \quad (38)$$

where last two lines are necessarily true, and thus further equivalent to

$$\begin{aligned} \exists \varphi \forall x \forall y ((fxy \wedge fyx) \vee \varphi x \vee \varphi y) & \quad \wedge \\ ((gxy \wedge hyx \wedge x \neq y) \vee \varphi x \vee \neg \varphi y) & \quad \wedge \\ ((hxy \wedge gyx \wedge x \neq y) \vee \neg \varphi x \vee \varphi y) & \quad \wedge \\ ((kxy \wedge kyx) \vee \neg \varphi x \vee \neg \varphi y). & \end{aligned} \quad (39)$$

Behmann arguments that the form (39) justifies to assume that in a problem expression of the form (37)

- the relations  $f$  and  $k$  are symmetric, and
- the relations  $g$  and  $h$  are irreflexive and inverse to each other.<sup>33</sup>

In his reply of 20 October 1934 [Letter L7], Ackermann notes that the component  $hxy \vee \neg \varphi x \vee \varphi y$  in (37) is not necessary, since it can be united with the component  $gxy \vee \varphi x \vee \neg \varphi y$  by switching variables. Accordingly, Behmann indicates in his manuscript [Manuscript M12] with pencil a second way of reading, where the component  $hxy \vee \neg \varphi x \vee \varphi y$  is dropped and  $g$  is just constrained by irreflexivity.

In [Manuscript M12], Behmann concludes the section with remarking that this was the point where he arrived several years before, but without seeing a prospect to make decisive progress. Before discussing Ackermann's result, he recapitulates the intermittent research on the decision problem for relations. He remarks that any reduction to the monadic case with equality is precluded, since a sentence like *There exists an infinite number of individuals* can for sure not be expressed by a sentence without predicates, such as, for example  $\exists x \exists y x \neq y$ . Hence, Bernays, Ackermann, Schönfinkel, and Schütte in Göttingen passed on to exploring with the much more primitive method of testing for particular domain cardinalities. The issue was to derive from the structure of the given sentence a suitable bound for the domain cardinality for which the test has to be made to get a result that applies to arbitrarily large domain cardinalities. However, this is not sufficient to manage a sentence that is as simple as *There exists an infinite number of individuals*, because tests can only be made for finite domain cardinalities, and even there only in theory and not practically. Behmann continues to summarize Schütte's result [Sch34, p. 603], which specifies for equality free sentences with quantifier prefix  $\forall \varphi_1 \dots \forall \varphi_h \forall x_1 \dots \forall x_k \exists y_1 \exists y_2 \forall z_1 \dots \forall z_l$  a number such that for all domain cardinalities that are larger or equal the truth value is the same. Behmann annotates that the result could have well been extended to consider also the case with equality. He proceeds to note that, in addition, Schütte has shown that for the case of more the two existential variables in the prefix such a number is no longer determinable in general. Behmann concludes this section with some acknowledging words on the effort and persistence of those working in these research directions, in awareness that the way would not lead to any practically usable results and that they would get stuck irrevocably at a rather early point.<sup>34</sup>

Behmann now turns in [Manuscript M12] to Ackermann's result in [Ack35a], noting that Ackermann's starting point was an example which was as simple as

<sup>33</sup> Although intuitively convincing, it seems still from a modern point of view not straightforward how a system that just gets (39) as input automatically utilizes these properties.

<sup>34</sup> *Es handelt sich bei den Bearbeitungen in diesen Forschungsrichtungen um sehr schwierige, aber mathematisch schöne und tiefe Untersuchungen. Und man muß die darauf verwandte Mühe und Ausdauer um so mehr bewundern, wenn man bedenkt, daß ja den Bearbeitern selbst nicht unbekannt sein konnte, daß sie auf diesem Wege überhaupt zu keinem irgendwie praktisch auswertbaren Ergebnis gelangen konnten und obendrein schon an einem ziemlich frühen Punkte endgültig stecken bleiben mußten.* [Manuscript M12], p.13f.

possible but for which he could show that the known means of representation failed. The axiom of induction (*Satz von der vollständigen Induktion*) can be expressed as the following second-order sentence

$$\forall\varphi(\varphi 0 \wedge (\forall m\forall n(\varphi m \wedge m + 1 = n) \rightarrow \varphi n)) \rightarrow \forall r \varphi r, \quad (40)$$

where domain of the individual quantifiers is the set of natural numbers. The quantifier  $\forall r$  can be moved in front of  $\forall\varphi$ . If the constant 0 and free variable  $r$  are written as  $a$  and  $b$ , respectively, and  $m + n = 1$  is abbreviated as  $fmn$ , then the elimination problem can be expressed as the first line in the following equivalences:

$$\begin{aligned} & \forall\varphi(\varphi a \wedge (\forall x\forall y(\varphi x \wedge fxy) \rightarrow \varphi y)) \rightarrow \varphi b \\ \equiv & \forall\varphi \neg\varphi a \vee (\exists x\exists y \varphi x \wedge fxy \wedge \neg\varphi y) \vee \varphi b \\ \equiv & \forall\varphi(\exists x x = a \wedge \neg\varphi x) \vee (\exists x\exists y \varphi x \wedge fxy \wedge \neg\varphi y) \vee (\exists y y = b \wedge \varphi y) \\ \equiv & \forall\varphi\exists x\exists y(y = a \wedge \neg\varphi x) \vee (\varphi x \wedge fxy \wedge \neg\varphi y) \vee (y = b \wedge \varphi y). \end{aligned} \quad (41)$$

The last two steps in (41) are obtained by EQ 33 and EQ 22, respectively. The dual or the negation of the last step matches the problem specification (37).

As Behmann recapitulates, Ackermann has *proven* in [Ack35a] that a predicate free formulation of the induction axiom is not possible, but Ackermann also says that in a certain new sense the elimination can nevertheless be performed. The induction axiom (in the form of the first line of (41)) expresses that one of the following sentences holds:

$$\begin{aligned} & a = b, \\ & fab, \\ & \exists x_1 fax_1 \wedge fx_1b, \\ & \exists x_1\exists x_2 fax_1 \wedge fx_1x_2 \wedge fx_2b, \\ & \dots \end{aligned} \quad (42)$$

That is, the induction axiom is equivalent to the disjunction of the infinite number of these sentences. With the notation  $f^0xy \leftrightarrow x = y$ ,  $f^1xy \leftrightarrow fxy$ ,  $f^2xy \leftrightarrow \exists z_1 fxz_1 \wedge fz_1y$ , etc., the resultant of the elimination can be written as

$$\exists n f^n ab, \quad (43)$$

where the domain of variable  $n$  is the set of natural numbers. The originally given sentence has then be brought into a form that is free from  $\varphi$ , contains on the one hand the numeric variable  $n$  as exponent, but, on the other hand, is very transparent.

Behmann proceeds that the question is now, whether this is not also possible in the general case, that is, for the formulas in (32). He says that Ackermann really succeeded there, giving a schema that generally characterizes the totality of the partial resultants of which the total resultant is disjunctively (or in the case  $\exists\varphi \dots$  conjunctively) composed. However, as Behmann remarks, on the basis of a *recursion* that is not quite easy to see through and thus lets the result appear somewhat intransparent. Refraining from summarizing Ackermann's result in

its original form, Behmann develops a presentation in a normalized form, as continuation of his own earlier work, starting from the normalized form (37), considered dually as:

$$\begin{aligned}
& \forall \varphi \exists x \exists y (fxy \wedge \varphi x \wedge \varphi y) & \vee \\
& (gxy \wedge \varphi x \wedge \neg \varphi y) & \vee \\
& (hxy \wedge \neg \varphi x \wedge \varphi y) & \vee \\
& (kxy \wedge \neg \varphi x \wedge \neg \varphi y), & \\
\end{aligned} \tag{44}$$

where  $fxy$ ,  $gxy$ ,  $hxy$  and  $kxy$  abbreviate  $F[\top, \top, x, y]$ ,  $F[\top, \perp, x, y]$ ,  $F[\perp, \top, x, y]$  and  $F[\perp, \perp, x, y]$ , respectively. The properties that can be assumed are symmetry of  $f$  and  $k$  as well as that  $g$  and  $h$  are inverse to each other and are reflexive (in the original: *totalreflexiv*), thus containing equality as sub-relation. As before (see p. 66), pencil annotations in [Manuscript M12] indicate a second way of reading, where the component  $hxy \vee \neg \varphi x \vee \varphi y$  is dropped. In [Manuscript M12] Behmann claims that his representation has for shown case of two individual variables the same (*sachliche*) generality as Ackermann's, who – Behmann seems to refer here to Ackermann's letter [Letter L7] – intentionally refrains from assuming that the given formula is normalized in the described way. He views the difference as only technical, where the preparatory work done by the normalization allows to express the resultant in a discernible more transparent way.

Following Behmann, the application of Ackermann's general recursive definition (*Rekursionsvorschrift*) to (44) yields a resultant of the form:

$$\begin{aligned}
& (\exists w \exists x \exists y \exists z \ gwx \wedge fxy \wedge hyz \wedge kzw \wedge \\
& \quad \neg \bigwedge_{x_1 \in \{w, x\}, x_2 \in \{y, z\}} x_1 \neq x_2) \\
\vee & (\exists s \exists t \exists u \exists v \exists w \exists x \exists y \exists z \ g(s, t) \wedge ftu \wedge huv \wedge kvw \wedge \\
& \quad gwx \wedge fxy \wedge hyz \wedge kzw \wedge \\
& \quad \neg \bigwedge_{x_1 \in \{s, t, w, x\}, x_2 \in \{u, v, y, z\}} x_1 \neq x_2) \\
\vee & \dots
\end{aligned} \tag{45}$$

In [Manuscript M12], a second variant of (45) without  $h$  is indicated in pencil:  $h$  is replaced there with  $\tilde{g}$ , the inverse of  $g$ .<sup>35</sup>

Before we continue with following Behmann's presentation, we take a closer look at his representation (45) of the resultant. For  $\bigwedge_{x_1 \in \{w, x\}, x_2 \in \{y, z\}} x_1 \neq x_2$  Behmann has the dedicated notation  $(wx, yz)$ . Expanding disequalities in the first disjunct of (45) shows that it stands for the disjunction of the following four (conjunctive) clauses:

$$\begin{aligned}
& (\exists w \exists x \exists z \ gwx \wedge fxw \wedge hwz \wedge kzw) \\
\vee & (\exists w \exists x \exists y \ gwx \wedge fxy \wedge hyw \wedge kwy) \\
\vee & (\exists w \exists x \exists z \ gwx \wedge fxx \wedge hxz \wedge kzw) \\
\vee & (\exists w \exists x \exists y \ gwx \wedge fxy \wedge hyx \wedge kxy).
\end{aligned} \tag{46}$$

<sup>35</sup> In the manuscript [Manuscript M12], but not in the letter [Letter L6], the variables  $z$  are universally quantified. Obviously a mistake in writing.

It is easy to see that some (conjunctive) clauses that do not contain  $\varphi$  can be obtained by (dual) resolution from (44), but are not subsumed by any disjunct of (45). For example

$$\exists x fxx \wedge kxx. \quad (47)$$

Thus, it seems that reflexivity of  $g$  and  $h$  needs to be considered such that also disjuncts obtained by removing one or more atoms with predicates  $g$  and  $h$  while unifying the left and right argument of each removed atom are considered as implicitly represented by (45).<sup>36</sup>

Behmann gives a second characterization of the resultant (45) as graph: "The resultant now means the following: If we consider the 4 relations  $f, g, h, k$  as arrow schemas (with different colors) inscribed into the same figure, then, according to the resultant, there is at least once a closed chain of arrows such that a  $g$ -arrow, an  $f$ -arrow, an  $h$ -arrow and a  $k$ -arrow follow once or a finite number of times cyclically in sequence, where, however, at least once in the chain a (starting or ending) point of a  $g$ -arrow coincides with a point of an  $h$ -arrow. I.e., there is at least one closed 8-course because of the cycle  $g, f, h, k$  with one or several rounds."

"If the symmetry conditions are omitted, that is,  $f, g, h, k$  are not submitted to any constraints, then this means for the chain of arrows stipulated by the resultant that the arrows from  $f, g, h, k$  may independently from each other be represented by  $\tilde{f}, \tilde{h}, \tilde{g}, \tilde{k}$ , and, moreover, the arrows from  $g$  and  $h$  may (again independently) be represented by identity (that is, by circular arrows), i.e. omitted in the chain."<sup>37</sup>

<sup>36</sup> Ackermann [Ack35a] provides a precise characterization of the clauses that can be obtained by resolution for his Example (21), a generalization of his Example (20), shown above as (28). Example (21) can be written dually as (44) with the two disjuncts containing  $g$  and  $h$ , respectively, dropped:

$$\begin{aligned} \forall \varphi \exists x \exists y (fxy \wedge \varphi x \wedge \varphi y) \quad \vee \\ (kxy \wedge \neg \varphi x \wedge \neg \varphi y). \end{aligned} \quad (48)$$

Thus, all (conjunctive) clauses that can be obtained by (dual) resolution from (48) must also be obtainable from (44). That is, all clauses satisfying Ackermann's characterization for his Example (21) should also be (in dual form) represented in Behmann's presentation of the resultant. The (conjunctive) clause (47) is an example.

<sup>37</sup> *Die Resultante besagt nun folgendes: Denken wir uns die 4 Beziehungen  $f, g, h, k$  als Pfeilschemata (mit verschiedenen Farben) in dieselbe Figur eingetragen, so gibt es gemäß der Resultante mindestens einmal eine geschlossene Kette von Pfeilen derart, daß ein  $g$ -Pfeil, ein  $f$ -Pfeil, ein  $h$ -Pfeil und ein  $k$ -Pfeil einmal oder endlich oft hintereinander zyklisch folgen, wobei aber mindestens einmal in der Kette ein (Anfangs- oder End-)Punkt eines  $g$ -Pfeils mit einem Punkt eines  $h$ -Pfeils zusammenfällt. D.h. es gibt mindestens eine geschlossene 8-Bahn auf Grund des Zyklus  $g, f, h, k$  mit einer oder mehreren Runden.*

Verzichtet man auf die Symmetriebedingungen, unterwirft man also  $f, g, h, k$  keiner Beschränkung, so besagt dies für die durch die Resultante geforderte Pfeilkette, daß die Pfeile aus  $f, g, h, k$  unabhängig von einander durch  $\tilde{f}, \tilde{h}, \tilde{g}, \tilde{k}$  vertreten sein

In the letter [Letter L6] to Ackermann, Behmann continues to explain his description and shows for Example (15) from [Ack35a] the resultant in his representation.<sup>38</sup> In our notation this is:

$$\begin{aligned}
&\forall x\forall y\forall z\forall u\forall v\forall p F[\lambda, \lambda, x, y] \vee \\
&\quad F[\lambda, \gamma, z, y] \vee \\
&\quad F[\gamma, \gamma, z, v] \vee \\
&\quad F[\lambda, \gamma, v, p] \vee \\
&\quad F[\lambda, \lambda, p, v] \vee \\
&\quad F[\lambda, \gamma, u, v] \vee \\
&\quad F[\gamma, \gamma, x, u],
\end{aligned} \tag{49}$$

where  $F[G, H, x_1, x_2]$  denotes  $F[\varphi x, \varphi y, x, y]$  with all occurrences of  $\varphi x$  replaced by  $G$ , of  $\varphi y$  by  $H$ , of  $x$  by  $x_1$  and of  $y$  by  $x_2$ .

In the manuscript [Manuscript M12], Behmann suggest that the cycle  $g, f, h, k$  can be better understood if it is already considered in the underlying problem specification, by writing (44) as:

$$\begin{aligned}
&\forall\varphi\exists x\exists y (gxy \wedge \varphi x \wedge \neg\varphi y) \vee \\
&\quad (fxy \wedge \varphi x \wedge \varphi y) \vee \\
&\quad (hxy \wedge \neg\varphi x \wedge \varphi y) \vee \\
&\quad (kxy \wedge \neg\varphi x \wedge \neg\varphi y),
\end{aligned} \tag{50}$$

It can then be seen that in the cycle

$$(\varphi x, \neg\varphi y), (\varphi x, \varphi y), (\neg\varphi x, \varphi y), (\neg\varphi x, \neg\varphi y),$$

adjacent components of neighboring pairs have complementary signs.

*dürfen und obendrein die Pfeile aus  $g$  und  $h$  (wiederum unabhängig) durch die Identität (also durch Rückkehrpfeile) vertreten sein, d.h. in der Kette wegbleiben, dürfen.*

From [Manuscript M12], p. 18. Similarly in the letter [Letter L6]. The alternate reading of  $h$  as inverse of  $g$  is indicated in pencil at two places in the first paragraph: *Beziehungen  $f, g, k$  and  $g$ -Pfeil, ein  $f$ -Pfeil, ein  $\tilde{g}$ -Pfeil und ein  $k$ -Pfeil.* In the second paragraph, *d.h. in der Kette wegbleiben* has been added in pencil. In the version from the letter [Letter L6], the corresponding text ends with *durch die Identität vertreten sein, also ganz ausfallen dürfen.*

<sup>38</sup> *Schreibe ich für  $F_{\gamma\gamma xy}$  kurz  $(\gamma\gamma)$  usf., so besagt die erste Bedingung, daß die Pfeile stets so aneinander zu fügen sind, daß für irgend zwei benachbarte Symbole  $(pq)$  und  $(rs)$  die Aussagewerte  $p q$  und  $r$  entgegengesetzt sind, und die zweite Bedingung, daß die Kette derart zu einer 8 zusammengebogen ist, daß jeder der beiden Bogen der 8 für sich genommen von zwei gleichartigen Symbolen begrenzt ist. Man sieht deutlich, wie durch die schärfere Normierung der Aufgabe die Durchsichtigkeit und anschauliche Erfassbarkeit der Resultante wesentlich erhöht wird. Durch das Mitführen der Aussagewertzeichen  $\gamma$  und  $\lambda$  wird der Sinn des Hoch- und Tiefstellens der Argumente aufgeklärt und dieses zugleich entbehrlich gemacht. So lautet Ihr Beispiel (15):*

$$xyzuvp(\lambda\lambda xy)(\lambda\gamma zy)(\gamma\gamma zv)(\lambda\gamma vp)(\lambda\lambda pv)(\lambda\gamma uv)(\gamma\gamma xu),$$

*schematisch:  $(\lambda\lambda)(\lambda\gamma)(\gamma\gamma)?(\lambda\gamma)(\lambda\lambda)?(\lambda\gamma)(\gamma\gamma)$ , wo die Unterstreichung den Ersatz von  $h$  durch  $\tilde{g}$ , d.h. von  $F_{\gamma\lambda xy}$  durch  $F_{\lambda\gamma yx}$  und die Zeichen  $?$  die zu identifizierenden Variablen andeuten.* From [Letter L6], p. 2.



Behmann recalls in [Manuscript M12] that his presentation can not be found in that form in Ackermann's paper [Ack35a], but – seemingly referring to letters [Letter L6] and [Letter L7] – that he was told by Ackermann upon request that he had actually been aware of it, but did not show it explicitly just for the reason that he did not succeed in finding an analogous representation for the case of three or more individual quantifiers.

In his letter [Letter L7], Ackermann writes that he was well aware that for the case of two universal quantifiers a complete overview on the resultants can be obtained and that this result indeed formed the basis of his work. The first outcome that he found was the resultant of

$$\exists f \forall x \forall y (axy \vee fx \vee fy) \wedge (dxy \vee \neg fx \vee \neg fy)^{39} \quad (51)$$

in the clear form (*in der übersichtlichen Form*). (Actually this is Example (21) from [Ack35a], shown – up to different predicate names – already as (48) in footnote 36 on p. 69.) Ackermann then found that the resultant of

$$\begin{aligned} \exists f \forall x \forall y (axy \vee fx \vee fy) & \quad \wedge \\ (bxy \vee fx \vee \neg fy) & \quad \wedge \\ (dxy \vee \neg fx \vee \neg fy) & \end{aligned} \quad (52)$$

is not much different from the resultant of (51); the cycles and chains have just to be extended in a suitable way by inserting  $b$  and  $\tilde{b}$ . The general case with two variables can be brought into form (52), which has also been discussed in Behmann's letter – as Ackermann remarks, the conjunct  $cxy \vee \neg fx \vee fy$  considered in addition by Behmann can be united with the middle component (see also p. 66). Passing on to more universal quantifiers, Ackermann found that a corresponding normalization (*Normierung*) can be easily achieved. For example, for three universal quantifiers it is

$$\begin{aligned} \exists f \forall x \forall y \forall z (axy \vee fx \vee fy \vee fz) & \quad \wedge \\ (bxy \vee fx \vee fy \vee \neg fz) & \quad \wedge \\ (cxy \vee fx \vee \neg fy \vee \neg fz) & \quad \wedge \\ (dxy \vee \neg fx \vee \neg fy \vee \neg fz), & \end{aligned} \quad (53)$$

where certain symmetry conditions hold for  $a, b, c, d$ . However, this normalization did not help Ackermann in getting an overview on the resultants at three and more universal quantifiers, such that he had dropped it again and had to confine himself in the general case to the recursion method for forming the resultant. To prevent that the gained overview for two universal quantifiers goes by the board, Ackermann included Example (20) [Ack35a, p. 410], which admits a simple interpretation of meaning. At the discussion of the example, the clear resultant for (51) is given [Ack35a, p. 411].

<sup>39</sup> We use here the predicate names from Ackermann's letter, which differ from that used by Behmann, but the correspondence is easy to see. Symbolic notation and capitalization are coherent with the rest of this paper, different from that used by Ackermann.

Manuscript [Manuscript M12] concludes with relating Ackermann's result to the general decision problem: "Actually, we just can say that *under favorable conditions*, that is, in so far as the assumed preconditions on individual variables are satisfied, *the innermost eliminations* can be performed, but not yet the further ones, since we do have the first obtained resultants *not in closed form*. Thus, there are two issues to solve: 1. the liberation from the *condition for the individual quantifiers* and 2. the representation of the respective resultant in a *closed symbolic representation* that is suitable for further eliminations." Behmann suggests that the first issue can be addressed by Skolemization, with the difficulties already described in Sect. 26. On the second issue, he remarks that in the case where all predicate quantifiers are universal (or existential) and at the front of the sentence, the problem can be considered as question of validity of a sentence that is free of predicate quantifiers.

Part V

Register of Publications by  
Behmann and Documents in  
his Bequest that are Relevant  
to Second-Order Quantifier  
Elimination



## 28 Introduction to Part V

This part provides commented lists of Behmann’s publications and unpublished material from his bequest [BehNL], such as manuscripts and correspondences, as far as they are immediately relevant to the problem of second-order quantifier elimination.

Accounts of contributions by Behmann in a historic context have been given in Church’s book *Introduction to Mathematical Logic* [Chu56], papers by Mancosu [Man99] and Zach [Zac99], a more recent presentation by Zach [Zac07] and the edition [MZ15] of Behmann’s 1921 talk on the *Entscheidungsproblem*. In Craig’s paper on the history on elimination problems in logic [Cra08], Behmann’s work [Beh22a] is briefly mentioned.

A recent English biography of Behmann can be found in [MZ15]. The most comprehensive publication of biographic material is in a dedicated chapter in [SM02, pp. 105–170] (in German), which also contains a selection of texts by Behmann.<sup>40</sup> He is positioned between Cantor and Husserl in this compilation about philosophical thinking in Halle (Saale), where he was professor of mathematics from 1925 to 1945. The investigation [Ebe02] (in German) of the *Martin-Luther-Universität Halle-Wittenberg* during Nazism includes a short biography of Behmann, but no further references to him. Behmann is peripherally mentioned in the treatise about logics in Nazi Germany in the context of Gentzen’s life [Men01] (English translation: [Men07]). Behmann’s personal file [BehAUH] at *Martin-Luther-Universität Halle-Wittenberg* is preserved in the university’s archive in Halle. Excerpts from the personal file, which include an 8-page typescript dated 20 October 1945 by Behmann about his activities in the NSDAP (*Zusammenstellung der für die Beurteilung wesentlichen Tatsachen*), are published in [SM02].

Behmann’s scientific bequest [BehNL] is located in the department for autographs of *Staatsbibliothek zu Berlin*. It has been registered by Peter Bernhard and Christian Thiel [BT00], after a first registration by Gerrit Haas and Elke Stemmler [HS81]. So far, it seems that there are only three publications of material from the bequest: The aforementioned talk on the *Entscheidungsproblem* from 1921 [MZ15], the correspondence with Gödel (from 1931) with English translation and an introduction by Charles Parsons [Göd03, pp. 12–39], and, from [BehAUH], a report about Behmann’s participation at the 1937 Congress for the Unity of Science in Paris [SM02, p. 105–108].

## 29 Publications by Behmann Related to Second-Order Quantifier Elimination

This compilation lists the publications by Behmann with immediate relevance to second-order quantifier elimination. These are the Habilitation thesis [Beh22a]

<sup>40</sup> There are some errata in [SM02]: P. 109: The birth name of Behmann’s mother is *Knübel* (not *Kübel*). P. 111 and 123: The quoted letter by Runge is dated 28 February 1926 (not November).

with some related documentary publications and abstracts surrounding it, as well as the abstract [Beh27a] of Behmann’s 1926 talk on the decision problem for relations. In addition, a later work [Beh50; Beh51] on the solution problem (*Auflösungsproblem*), where techniques from [Beh22a] are applied, is listed. Also two further major publications [Sch24; Beh27b] by Behmann, or with involvement of Behmann, respectively, are included, because they fall in the time span between 1921 and 1927, although they are not directly concerned with elimination or the decision problem.

As already noted on p. 8, Behmann’s extensive investigations of the paradoxes are not considered here, although one may speculate whether his underlying idea that paradoxes emerge from unjustified elimination of shorthands is somehow related to elimination of second-order quantifiers.

- [Guz21] (1921) The annual report of the *Deutsche Mathematiker-Vereinigung* shows in a listing of the activities of the *Mathematische Gesellschaft in Göttingen* that Behmann gave on 10 May 1921 at the *Mathematische Gesellschaft* a talk *Das Entscheidungsproblem der mathematischen Logik*, which can be translated as *The Decision Problem of Mathematical Logic*. As noted in [Zac99, p. 363], this seems the first documented use of *Entscheidungsproblem*. The manuscript [Manuscript M1] *Entscheidungsproblem und Algebra der Logik*, published recently in [MZ15], bears the date of the talk and thus seems to underlie it.
- [Beh22a] (1922, received by the journal on 16 July 1921) Heinrich Behmann: *Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem*. This is the published version of the thesis for Behmann’s Habilitation at Göttingen on 9 July 1921. Some corrections of printing errors have been published as [Beh22b]. A carbon copy of the thesis typescript with handwritten corrections that have been considered for the printed version is preserved as [Manuscript M3]. Document [Manuscript M4] is an author’s offprint of the published version with later handwritten corrections.
- [Beh22b] (1922) *Druckfehlerberichtigung zu dem Aufsatz von H. Behmann „Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem“ in Band 96, S.163–239*. 1922. Corrections of nine wrongly printed symbols in [Beh22a].
- [Beh23] (1923) Heinrich Behmann: *Algebra der Logik und Entscheidungsproblem*. Abstract of a talk given on 21 September 1923 at the *Jahresversammlung der Deutschen Mathematiker-Vereinigung* in Marburg a. d. Lahn. A manuscript for the abstract is preserved as [Manuscript M7]. The talk seems in essence a summary of the results published in [Beh22a]. In the subsequent discussion, L. E. J. Brouwer (who gave a talk in the same session) expressed concerns about the phrase *usw.* (*and so on*) and the concept of finite number within the presented theory. These concerns have been rebutted by remarks that

the debated concepts do not play a role on their own in the considered statements but can there be reduced unobjectionable to basic concepts and that the theory about these statements is just a theory within mathematics, not intended as foundation of mathematics.

- [Sch24] (1924, received by the journal on 15 March 1924) Moses Schönfinkel: *Über die Bausteine der mathematischen Logik*. In this paper logic combinators have been introduced. As noted in the paper, it is based on a talk by Schönfinkel given in 1920 at the *Mathematische Gesellschaft in Göttingen* and has been prepared for publication and supplemented by Behmann. William Craig asks Behmann in a letter dated 16 March 1952, upon advice of Haskell Curry, about an error in the supplementary part by Behmann. In his reply dated 7 April 1952, Behmann remarks that he did the preparation for publication on behalf of Hilbert, without a particularly strong interest and that he considers the new direction that emerged from that work as too formalistic. He was aware of the error, which had been pointed out to him in 1928 by Alfred Boskowitz [BehNL, Kasten 1, I 13 and I 14].<sup>41</sup> See also [CFC58, p. 8, p. 184].
- [Beh27a] (1927). Heinrich Behmann: *Entscheidungsproblem und Logik der Beziehungen*. Abstract of a talk given on 23 September 1926 at the *Jahresversammlung der Deutschen Mathematiker-Vereinigung* in Düsseldorf. This abstract motivated Ackermann to write to Behmann in 1928 [Letter L1], which initiated their correspondence on elimination for relations. A draft manuscript is preserved as [Manuscript M11].
- [Beh27b] (1927) Heinrich Behmann: *Mathematik und Logik*. A small introductory textbook on mathematical logic, showing Gottlob Frege on the cover. The twofold connection between mathematics and logics is emphasized: Mathematical representation and notation allows to make logic possible as an “exact” science, like mathematics, and, on

<sup>41</sup> Alfred Boskowitz was a student in Göttingen when Behmann was lecturer. In the mid 1920s Boskowitz moved back to Budapest, his home town. He carefully reviewed and extended the *Principia Mathematica*, where he is mentioned in the second edition in an acknowledging footnote together with Behmann [WR27, p. xiii]. Respective manuscripts by Boskowitz are in [BehNL, Kasten 1, I 08]. Behmann uses the opportunity of Curry’s request for information about Boskowitz [BehNL, Kasten 1, I 08] on 12 July 1957 to draw attention to Boskowitz’s work and sends to Curry on 13 August 1957 [BehNL, Kasten 1, I 08] a characterization of Boskowitz as well as typed transcripts of a selection of his letters until 1937. Those letters and transcripts are now in [BehNL, Kasten 1, I 08]. On 21 June 1936 Boskowitz writes to Behmann that he expects danger of life and asks Behmann to store his mathematical works. In his letter to Curry, Behmann states that he did not hear from Boskowitz since 1937 and the address in Budapest from that time to Curry. However, in [BehNL, Kasten 1, I 08] there is also a short letter from 1939 by Boskowitz as well as a postcard dated 11 November 1942 with a different address in Budapest.

the other hand, the insight that, conversely, pure mathematics is nothing else than logic in disguise.

[Beh50] (1950/51). Heinrich Behmann: *Das Auflösungsproblem in der Klas-*  
 [Beh51] *senlogik*. Behmann develops an approach to solve the “logic solution problem” (*logisches Auflösungsproblem*). In a somewhat different phrasing than by Behmann, it can be described as follows: Given is a formula where the set of predicates occurring in it is partitioned into two disjoint subsets. The objective of the solution problem is to find a representation of the relation of predicate valuations from the first subset to those valuations from the second subset for which the formula is true. In addition, conditions on the predicates from the first subset are sought, under which solution valuations for those from the second subset exist at all.

Behmann relates the solution problem to the elimination problem, discusses earlier works by Schröder, Jevons and Boole and extends Boole’s approach to  $\text{MON}_=$ . Behmann’s method is based on a normalization: Assume that the predicates under consideration are partitioned into  $\{p_1, \dots, p_n\}$  and  $\{p\}$ . A  $\text{MON}$  formula without individual constants over these predicates can be converted into a disjunction of conjunctions in which each conjunct has one of the following basic forms:

$$\begin{aligned} \text{(a)} \quad & \neg\exists x L_1[x] \wedge \dots \wedge L_n[x] \wedge px, \\ \text{(b)} \quad & \neg\exists x L_1[x] \wedge \dots \wedge L_n[x] \wedge \neg px, \\ \text{(c)} \quad & \exists x L_1[x] \wedge \dots \wedge L_n[x] \wedge px, \\ \text{(d)} \quad & \exists x L_1[x] \wedge \dots \wedge L_n[x] \wedge \neg px, \end{aligned} \tag{54}$$

where each  $L_i[x]$  is either  $p_i x$  or  $\neg p_i x$ . This form can be obtained with the techniques used in [Beh22a] for elimination, followed by conversion to the “fully developed” form, where each basic formula contains exactly one literal with each predicate in  $p_1, \dots, p_n, p$ . The fully developed form can be achieved by rewriting with

$$\neg\exists x F[x] \equiv (\neg\exists x F[x] \wedge qx) \wedge (\neg\exists x F[x] \wedge \neg qx) \tag{55}$$

and

$$\exists x F[x] \equiv (\exists x F[x] \wedge qx) \vee (\exists x F[x] \wedge \neg qx). \tag{56}$$

Without loss of generality, it can be assumed that the conjunctions of formulas of forms in (54) do not contain contradicting conjuncts. Such a conjunction then corresponds to a solution set, represented by the mapping from the the  $2^n$  formulas  $L_1[x] \wedge \dots \wedge L_n[x]$ , where each  $L_i[x]$  is either  $p_i x$  or  $\neg p_i x$  to one of nine values depending on which of the forms (a)–(d) containing  $L_1[x] \wedge \dots \wedge L_n[x]$  are present. For given  $L_1[x] \wedge \dots \wedge L_n[x]$ , there are nine such combinations of (a)–(d) whose conjunction is not contradicting. The total solution is a set of such mappings, one for each disjunct of the normalized formula.



### 30 Manuscripts and Other Archival Documents by Behmann Related to Second-Order Quantifier Elimination

This section provides a commented listing of the manuscripts and other archival documents in Behmann's scientific bequest [BehNL] that are related to second-order quantifier elimination, in chronological order. Of these, the manuscripts [Manuscript M1], [Manuscript M6], and [Manuscript M10] were in the bequest originally in the so-called "brown box" (*braune Box*), a cuboid cardboard folding box covered with brown-black marbled paper with a handwritten note by Behmann "*Important records for own lectures in Göttingen and Halle! H. Behmann*" ("*Wichtige Aufzeichnungen für eigene Vorlesungen in Göttingen und Halle! H. Behmann*") [BT00, p.2, p. 78] which included what is now registered as Kasten 9, Einheit 1–45 and Kasten 10, Einheit 46–91 of [BehNL].

Aside of the individual manuscripts listed below, further documents that concern the elimination and decision problem can be found in [BehNL, Kasten 11] which contains among other documents about 200 pages of notes that are not registered in [BT00]. Most of these notes are inscribed almost purely with formulas, occasionally with graphical visualizations. Some of them evidently concern the variant of Ackermann's resolution-based elimination method as presented in 1934 by Behmann in manuscript [Manuscript M12] and letter [Letter L6], which is summarized in Sect. 27. For others it may be conjectured that they relate to the elimination and decision problem for relations and to Skolemization. A particular ordering of the notes is not immediate. Some of them bear explicit dates, including particular days in February, March and April 1924, July 1926, March 1928 and August 1934 (some are just dated *August 1934*, without specification of a day). The respective folders in [BehNL, Kasten 11] where they are contained are *hsl. Aufzeichnungen mathematischer u. physikalischer Art* and *Logik I, Logik II hsl. Aufzeichnungen*.

- M1 *Entscheidungsproblem und Algebra der Logik*. 1921. In [BehNL, Kasten 9, Einheit 37], see [BT00, p. 88]. Handwritten manuscript. 17 numbered pages. Dated 10 May 1921. Written in ink with red underlines and side notes, corrections and addenda in pencil, some in shorthand. Starts after the title with "*Von Kronecker stammt, soviel ich weiß, ein Distichon, das ungefähr so lautet:*".

Summarizes the material in [Beh22a]. Seems to be the manuscript for the talk on 10 May 1921 at *Mathematische Gesellschaft in Göttingen* listed in [Guz21] as *Das Entscheidungsproblem der mathematischen Logik*. A transcript has been published along with an introduction and English translation as [MZ15].

- M2 *Das Problem der Axiomatik vom Standpunkt der Algebra der Logik*. 1922. In [BehNL, Kasten 9, Einheit 33], see [BT00, p. 87]. Handwritten manuscript. Dated 28 November 1922. 4 pages on 1 folded double sheet. Starts after the title with "*Bemerkungen zu dem Vortrag von H. Neues und bemerkenswertes Problem aufgeworfen. Allerdings recht speziell.*"

- M3 *Beiträge zur Algebra der Logik, insbesondere zum Entscheidungsproblem*. 1921. In [BehNL, Kasten 7, 1922], see [BT00, p. 61]. Typescript (carbon copy) of the thesis of Behmann’s Habilitation at Universität Göttingen with handwritten mathematical symbols and corrections that have been considered in the published version [Beh22a]. 79 numbered pages plus two initial pages for title and table of contents.
- M4 Author’s offprint of [Beh22a]. 1922. In [BehNL, Kasten 6, 1921.1], see [BT00, p. 110]. With handwritten notes and corrections. On top of the title page a handwritten dedication in English “*With the author’s compliments.*” and fragments of a postmark.
- M5 *Algebra der Logik und Entscheidungsproblem*. 1923?. In [BehNL, Kasten 6, V 23], see [BT00, p. 53, also p. 44]. Handwritten manuscript. 39 pages in a checkered notebook containing also other texts on other topics.

The notebook also contains various other long and short texts, some seemingly related to lectures given by Behmann, such as *Darstellende Geometrie*. Further texts include drafts for the letter dated 10 May 1923 to Bertrand Russell [BehNL, Kasten 2, I 60] and an excerpt from the article on *Calculating Machines* in *Encyclopedia Britannica*.<sup>42</sup>

The manuscript *Algebra der Logik und Entscheidungsproblem* starts after the title with “*Vortragstext (stellenweise weiter ausgeführt) bis [bei?] S. 4*”. A paragraph header “*Aufgabe der Vorlesung*” on the first page suggest that this text is a draft manuscript for a lecture.

- M6 *Algebra der Logik und Entscheidungsproblem*. 1923/1924. In [BehNL, Kasten 9, Einheit 45], see [BT00, p. 91]. Handwritten manuscript. 60 numbered sheets, with p. 6, p. 32–35, p. 47, p. 50, p. 57 inscribed also recto. One additional sheet inserted after the page 1. Ink with pencil additions. Dated “*W. S. 1923–24*”. Starts after the title with “*Kronecker (Mathematiker), Distichon:*”

Seems to be the manuscript for Behmann’s lecture with the same title listed in the register of lectures at Göttingen university for winter semester 1923/24 (*Verzeichnis der Vorlesungen der Universität Göttingen Winterhalbjahr 1923/24*).<sup>43</sup>

<sup>42</sup> The excerpt corresponds in the online edition of the 1911 Encyclopedia Britannica to the span from “*Machines of far greater powers*” to “*published by his son, General Babbage.*”. See [http://en.wikisource.org/wiki/1911\\_Encyclop%C3%A6dia\\_Britannica/Calculating\\_Machines](http://en.wikisource.org/wiki/1911_Encyclop%C3%A6dia_Britannica/Calculating_Machines) (accessed 17 August 2015). Behmann also annotates the author of the article as “*Henrici*”, with “*1792–1871*” added in pencil. In fact, the author was Olaus Magnus Friedrich Erdmann Henrici (1840–1918).

<sup>43</sup> Behmann regularly gave lectures in mathematics, 1921–26 as doctor in Göttingen and 1926–45 as professor in Halle. Only very few of these were on logic or related topics: WS 1921/22 Set Theory, SS 1922 Mathematical Logic (Also for Non-Mathematicians), WS 1923/24 Algebra of Logic and Decision Problem, SS 1927 Mathematics and Logic.

In [BT00] it is conjectured that the untitled 30-page document [BehNL, Kasten 10, Einheit 90] might be a continuation of [Manuscript M6]. However, consideration of three-valued logic and notation of existential quantifiers as  $\exists$  in [BehNL, Kasten 10, Einheit 90] suggests that it stems from a later period.

M7 *Entscheidungsproblem und Algebra der Logik*. 1923. In [BehNL, Kasten 7, 1923], see [BT00, p. 61]. Typescript (carbon copy) with handwritten corrections. 3 pages. Manuscript for the talk abstract [Beh23].

M8 *Entscheidungsproblem für Beziehungen*. 1926?. In [BehNL, Kasten 8, 1925/26.1], see [BT00, p. 58]. Handwritten draft. 4 pages. Starts after the title with “*Wir haben als Bestandteil einer Aussage den folgenden:*”.

M9 *Entscheidungsproblem und Logik der Beziehungen*. 1926?. In [BehNL, Kasten 8, 1925/26.2], see [BT00, p. 58]. Handwritten draft. 4 pages. Starts after the title with “*Wer über ein Problem der mathematischen Logik vorzutragen gedenkt*”.

Seems to be a draft for the talk on 23 September 1926 in Düsseldorf, whose abstract has been published as [Beh27a]. The document [Manuscript M10] seems to be a later but incomplete version.

M10 *Entscheidungsproblem und Logik der Beziehungen*. 1926?. In [BehNL, Kasten 10, Einheit 46], see [BT00, p. 91, also p. 57]. Handwritten manuscript. 11 pages (9 numbered sheets with text on verso only, 1 sheet with formulas inscribed verso and recto). Filed together with an additional seemingly unrelated double sheet that is inscribed on 3 pages. Starts after the title with “*Wer die Absicht hat, über ein Thema aus dem Gebiet der mathematischen Logik vorzutragen*”.

Seems to be a later but incomplete version of [Manuscript M9]. The text on page 9 ends abruptly, where [Manuscript M9] continues after the matching position. As noted in Sect. 31 about [Letter L4], Behmann has sent in 1928 the technical second part of an elaborate transcript of his talk on 23 September 1926 in Düsseldorf to Ackermann, which suggests that this incomplete manuscript might be the first part of this transcript. The later manuscript [Manuscript M12] also includes a presentation of Behmann’s work on elimination for relations around 1926.

M11 *Entscheidungsproblem und Logik der Beziehungen*. 1926. In [BehNL, Kasten 8, VII 01–VII 06]. Not registered in [BT00] (see [BT00, p. 61]). The document is the last one in the aforementioned folder. Handwritten draft manuscript of the talk abstract published as [Beh27a]. 2 pages on a 4 page double sheet, along with a draft of a short letter from Rome, dated 23 October 1926, to accompany the submission of the manuscript and parts of an unrelated manuscript. The draft letter is addressed to a professor, probably Bieberbach as editor, but does not contain any discussion regarding content (compare p. 60).

The following two portions of the draft manuscript do not appear in the published version: (1.) “[. . . Elimination in der Tat vollziehen läßt.] So hat das erste die bekannte Lösung  $x(\bar{p})F_{xp}$  bzw.  $x F_{\lambda x} F_{\gamma x}$ , während z.B. für das zweite [XXX] die Lösung

$$x(\bar{p}) y(\bar{q}) F_{pqxy} \cdot y(\bar{q}) x(\bar{p}) F_{pqxy}$$

ermittelt wurde. [So lange . . .]”. (Transcript with the symbolic notation in the manuscript. [XXX] marks a word that could not be identified.) (2.) “[. . . auch verwickeltere Verkettungen] dieser Operationen, insbesondere Zyklen, die etwa nicht mehr der Transitivität des Früher oder Später genügen, [zuzulassen. . .].” Behmann uses in this text “Dingoperator” for *quantified variable*, “Begriffoperator” for *quantified predicate* and “Operation” for *quantifier*. In the published version, the overline indicating existential quantification has been erroneously omitted at some occurrences of  $\varphi$ : two occurrences in the displayed row of formulas in the center of p. 17 and in the first line of the last paragraph of p. 17. This manuscript is further discussed in Sect. 25 and 26.

- M12 *Ein wichtiger Fortschritt im Entscheidungsproblem der Mathematischen Logik (Ackermann Math. Annalen 110 S.390)*. 1934. In [BehNL, Kasten 8, 1934], see [BT00, p. 59]. Handwritten manuscript. Dated 14 December 1934. 21 numbered pages. Written in ink with some variants and corrections added in pencil.

For a summary, see Sect. 27. The technical part concerning the elimination problem for relations overlaps with [Letter L6]. A few phrases could be interpreted as suggesting that the manuscript was intended as basis for a talk (on p. 17: “*Ich möchte darauf verzichten, [. . .] in der von ihm gegebenen Allgemeinheit vorzutragen*”, on p. 19: “*Sie werden nun genauer wissen wollen*”).

### 31 The Correspondence between Heinrich Behmann and Wilhelm Ackermann

The complete correspondence between Heinrich Behmann and Wilhelm Ackermann, as far as preserved in [BehNL, Kasten 1, I 01], see [BT00, p. 3], is listed here with English abstracts. Some technical content beyond second-order quantifier elimination is also summarized, however, the discussion of Behmann’s ideas for the resolution of the paradoxes and related issues such as restricted (*limitierte*) variables and ultrafinitistic logic is only briefly indicated here, since it would form a major topic on its own.

The letters by Behmann archived in [BehNL] are handwritten copies or carbon copies of typescripts, respectively. Seemingly, there is an erratum in [BT00]: Instead of the two letters from Ackermann to Behmann dated 29 October 1934 [Letter L7] and 9 January 1953 [Letter L8], a single letter dated *29 October 1953*, not present in the bequest, is listed there.

Upon request by Christian Thiel, Ackermann's son Hans-Richard Ackermann writes on 23 August 1981 [BehNL, Kasten 12] that, as far as Behmann is concerned, in a first sighting of his father's correspondence, he could only find a carbon copy of the letter to Behmann from 29 October 1934 [Letter L7], whose original is already present in Behmann's bequest. In the selection from the correspondence of Wilhelm Ackermann [Ack83] published in 1983 by Hans-Richard Ackermann, the correspondence with Paul Bernays concerns the elimination problem (see p. 62).

For a general introduction to the correspondence between Behmann and Ackermann as well as technical summaries of the content regarding second-order quantifier elimination see Part IV.

L1 Ackermann to Behmann, Lüdenscheid, 16 August 1928, handwritten, 2 pages.

See Sect. 25.

L2 Behmann to Ackermann, Nieblum auf Föhr, 21 August 1928, handwritten, 2 pages. In addition to answering [Letter L1], this letter refers to an unidentified preceding mail from Ackermann with the copy of the book [HA28].

See Sect. 25 for the content regarding elimination. Behmann wishes Ackermann success on this way, emphasizing the importance of the decision problem for strengthening the reputation of symbolic logic in wider circles, mentioning Heinrich Scholz, who had to experience combating against symbolic logic with unpleasant means. In addition, Behmann thanks Ackermann for transmitting a copy of his book [HA28] and indicates that he does not agree with all details, in particular regarding the Hilbert-Bernays symbolism, but admires it as a whole.

L3 Ackermann to Behmann, Münster i. W., 1 September 1928, handwritten, 2 pages.

See Sect. 25 for the content regarding elimination. Ackermann further reports that he just sent off the corrections for his paper [Ack28] (decidability of the class called today *Ackermann class*), motivated by the work of Bernays and Schönfinkel [BS28], who, in Ackermann's view, take too much effort for simple special cases. Ackermann requests more information about the details in his book [HA28] with which Behmann disagrees (see [Letter L2]). In a possible second edition some changes would be made. Among other things, Ackermann intends to include a presentation of the results of Behmann's Habilitation thesis [Beh22a].

L4 Behmann to Ackermann, Halle (Saale), 29 September 1928, typescript, 2 pages. The letter originally enclosed two manuscripts by Behmann.

Behmann writes that he had not found the time to compile his results on the decision problem in a well arranged way (as he had announced in [Letter L3], see also p. 60). Thus he encloses the transcript of his talk in Düsseldorf, which, he writes, is much more elaborated than the talk itself

could be. He sends only the part that would be interesting to Ackermann, omitting the introductory sections. What turned out incorrect is with red pencil put into parentheses or struck out. Behmann writes that there is no rush to return the transcript, if he would need it in foreseeable time, he would give notice.

So far, the part of the transcript sent to Ackermann could not be located in Behmann's bequest. Possibly [Manuscript M10] is the first part that retained with Behmann and [Manuscript M9] is an early version.

Another topic discussed at length in the letter is Behmann's work on the resolution of the paradoxes. He encloses a carbon copy of a manuscript about this, which he had sent to Hilbert a few days before.<sup>44</sup> Behmann concludes the letter with announcing that he will send his comments on the details in the book [HA28] (see [Letter L3]) as soon as he has looked through it for these.

L5 Ackermann to Behmann, Münster i. W., 1 November 1928, handwritten, 4 pages.

See Sect. 25 and 26 for the content regarding elimination. Ackermann further considers the question of what formulas can be proven and what can not be proven from the axioms on p. 53 in [HA28]. He sketches an idea for a method to verify formulas that are valid for all domains which operates by successively strengthening formulas by melting quantifiers until their matrix is propositionally valid. He gives two examples, presented here in tabular form and modern notation:

$$\begin{aligned}
 (1.) \quad & \forall x \exists y fxx \vee \neg fxy \\
 (2.) \quad & \models \forall x fxx \vee \neg fxx \\
 (3.) \quad & \equiv \top.
 \end{aligned} \tag{57}$$

$$\begin{aligned}
 (1.) \quad & \exists x \forall y fxx \vee \neg fyy \vee fxy \\
 (2.) \quad & \equiv (\exists x \forall y fxx \vee \neg fyy \vee fxy) \vee (\exists u \forall v fuu \vee \neg fvv \vee fuv) \\
 (3.) \quad & \equiv \exists x \forall y \exists u \forall v fxx \vee \neg fyy \vee fxy \vee fuu \vee \neg fvv \vee fuv \\
 (4.) \quad & \models \exists x \forall y \forall v fxx \vee \neg fyy \vee fxy \vee fyy \vee \neg fvv \vee fyv \\
 (5.) \quad & \equiv \top.
 \end{aligned} \tag{58}$$

Step (2.) of (58) is formed by disjoining (called “*multiplizieren*” by Ackermann) the given expression with itself. Step (4.) is obtained by melting  $\forall y$  and  $\exists u$ , that is, the  $\exists u$  is “sucked up” by the preceding universal quantifier.

<sup>44</sup> There are just two letters from Behmann to Hilbert in Behmann's bequest [BehNL], carbon copies of typescripts dated 18 September 1928 and 25 September 1928 [BehNL, Kasten I 32], both of them with technical discussions of paradoxes, the latter with corrections to the first. Behmann also sent these on 29 September 1928 to Frank P. Ramsey, who gave a detailed reply on 4 October 1928, leading to a further letter by Behmann dated 9 October and by Ramsey dated 16 October [BehNL, pp. I 56]. Behmann refers briefly to that correspondence in [Beh37a, p. 220].

The difficulty is, according to Ackermann, that an expression has to be first disjoined with itself a finite number of times before the finite number of possible variable meltings can be investigated, and the number of disjoined copies has to be set in advance. Ackermann notes that the method is certainly insufficient for formulas with quantifier prefixes

$$\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m$$

and

$$\forall x_1 \dots \forall x_n \exists y \exists z_1 \dots \exists z_m,$$

but that so-far he could not provide a general proof.

The letter further discusses Behmann's idea about the resolution of the paradoxes (see [Letter L4]). Ackermann writes that some issues did not become fully clear to him and gives an example to clarify.

- L6 Behmann to Ackermann, Halle (Saale), 22 October 1934, typescript with handwritten insertions at formulas, 3 pages. Behmann sends at the same time some offprints. Reference to an unidentified previous mail in which Ackermann had sent an offprint of the paper [Ack35a] to Behmann.

See Sect. 27 for the content regarding elimination. Behmann gives some remarks on notation and presentation: He suggests to replace the confusing terms “*Summe*” and “*Produkt*” by “*Konjunktion*” and “*Disjunktion*” and recommends to call  $f, g, h, k$  constants instead of variables, referring to [Beh22a, footnote 25, p. 196] (see also Sect. 10.4). In a postscript he proposes to proceed with simplifying Hilbert's symbolism by removing *everything* that is dispensable until the differences to Behmann's notation are no longer worth mentioning. In particular, the large extent of the formulas and the clutter of the many parentheses (“*das Gestrüpp der vielen Klammern*”) are considered by Behmann as quite perturbing at practical work.

Behmann writes that references to sources, wherever appropriate, would be desirable, giving [Sch95, pp. 512–516] for p. 412 (which actually concerns Skolemization) as an example. In particular, he does not see to what extent Ackermann has made explicit use of Behmann's earlier communications (in relation to his talk in Düsseldorf), or came to the same findings already independently.

Behmann concludes with asking Ackermann to send an offprint of [Letter L6] to Boskovitz<sup>45</sup>

This letter is the only one in the Behmann-Ackermann correspondence in [BehNL] that ends with a Nazi salutation, *mit deutschem Gruß*, common in Germany at that time for official letters [Ehl12].

<sup>45</sup> Alfred Boskovitz, see footnote 41 on p. 77.

L7 Ackermann to Behmann, Burgsteinfurt, 29 October 1934, typescript with handwritten formulas, 5 pages.

See Sect. 27 and also 26 for the content regarding elimination. Ackermann replies to Behmann’s points of criticism, kindly remarking that, as can already be seen at the formulas in the letter, he has adopted Behmann’s notation for the universal and existential quantifiers for his own practical work, and referring to Behmann’s small book [Letter L7] as a rich source in this respect.<sup>46</sup> However, he considered himself as bound to the notation he and Hilbert used in their introduction to logic [HA28].

Schröder’s idea underlying the “*Belegungsfunktionen*” (known today as Skolem functions) has, according to Ackermann, become such common knowledge of logicians that in his view there would be no need to cite Schröder. As an example he refers to Skolem’s use and simplified notation in 1920. He writes that he had investigated the decision problem and elimination problem in the case where Skolem functions do occur already in 1925: he found an excerpt of [Sch95] from that time where he has proven for nine of ten resultants that have in part just been conjectured by Schröder and brought somehow into connection with Peirce (Ackermann is only in possession of this excerpt of Schröder’s book) the correctness of Schröder’s conjectures. The method is the same as Example (26) in [Ack35a], which actually belongs to the mentioned ones by Schröder. Thus, Ackermann believes to be independent of Behmann’s communications in his remarks in Section 6 of [Ack35a] (where existential quantifiers are discussed), the only section to which Behmann’s comment in [Letter L6] might refer. Ackermann notes that he obviously could have mentioned Behmann’s talk in Düsseldorf and their correspondence that followed. He writes that the large temporal distance might be an excuse and announces to make up for it at the next occasion.

Skolem functions are no longer considered as an advantage by Ackermann (see Sect. 23). His alternative approach is elaborated later in [Ack35b] (also see Sect. 23), where, in a footnote he explains – related to the suggestions by Behmann in [Letter L6] – the history of Skolemization and mentions that Behmann has brought the advantages of the introduction of Skolem functions for the elimination problem to attention already in his talk at the *Mathematikerversammlung* in Düsseldorf 1926.

Concerning Behmann’s ultrafinitist logic (seemingly the topic on an offprint or manuscript sent by Behmann with [Letter L6]), Ackermann urges him to elaborate a precise formulation and brings the evidently similar theory by Church [Chu32; Chu33] to attention.

<sup>46</sup> *Was die Symbolik anbetrifft, so sehen Sie schon an den hingeschriebenen Formeln, dass ich mir für meine praktischen Arbeiten auch Ihre Schreibweise für das All- und das Seinszeichen angeeignet habe, wie überhaupt Ihr Büchlein über “Mathematik und Logik” eine wahre Fundgrube in dieser Beziehung ist.*



Ackermann confirms that he had sent an offprint of [Ack35a] to Boskovitz (see [Letter L6]). In a postcard to Behmann dated 17 June 1935 [BehNL, Kasten 1, I 08], Boskovitz writes that he received the offprint in fall and thanks Behmann, conjecturing that the address and probably the idea are due to him. However, technical aspects of [Ack35a] are not discussed by Boskovitz.

The letter concludes with Ackermann remarking that he has heard from Arnold Schmidt<sup>47</sup> that Behmann had been on the congress in Pymont (*Jahrestagung der Deutschen Mathematiker-Vereinigung 1934*) and thus regrets to have dropped his original intention to also go there.

- L8 Ackermann to Behmann, Lüdenscheid, 9 January 1953, typescript, 1 page. Reference to an unidentified previous postcard and mail by Behmann, where Behmann sent a copy of his treatise “*Deskription und limitierte Variable*” [Beh52] and requested the address of Sören Halldén<sup>48</sup>.

Ackermann thanks Behmann for his treatise, and writes that it had interested him very much – as well as a continuation (*Weiterführung*) would do. He cites it later in [Ack58] with just *1944 and extended 1952* as bibliographic details. He does not know the full address of Sören Halldén but says that mails with just *Universität Uppsala* as address arrived. As American experts that would be interested in [Beh52], he mentions Church and Quine.

- L9 Ackermann to Behmann, Lüdenscheid, 22 October 1954, handwritten, 2 pages. Reference to a previous mail by Behmann with copies of the treatises *Zur Technik des Schließens und Beweisens* [Beh54b] and *Ein neuer Vorschlag für eine einheitliche logische Symbolik* [Beh54a].

Ackermann discusses the standardization of logic symbolism, referencing also to Behmann’s earlier works on the subject, one from 1935 and [Beh37b]. He suggests that Behmann should submit [Beh54a] to the *Journal of Symbolic Logic*.

- L10 Behmann to Ackermann, Bremen-Aumund, 6 October 1955, typescript, 2 pages. Carbon copies of a draft manuscript *Die Besetzungskette und der widerspruchsfreie Prädikatenkalkül* are originally enclosed.

Planning to submit a revision of his manuscript to the *Journal of Symbolic Logic* (which was published there indeed as [Beh59]), Behmann asks Ackermann for his judgment regarding content as well as form, and to forward copies to Hermes, Scholz and probably Hasenjaeger.<sup>49</sup> Behmann intends to visit Münster for a week and give a talk. He stresses the importance of the problem of a contradiction-free predicate calculus. He would have also sent copies of his talk [Beh55], which he had submitted but not given, but the

<sup>47</sup> Hermann Arnold Schmidt (1902–1967), German mathematician.

<sup>48</sup> Sören Halldén (1923–2010), Swedish philosopher and logician.

proceedings did not yet appear. Finally, he asks Ackermann again about his judgment on the previously sent treatise [Beh54b].

- L11 Ackermann to Behmann, Lüdenscheid, 27 October 1955, typescript, 3 pages.

Ackermann discusses Behmann's manuscript *Die Besetzungskette und der widerspruchsfreie Prädikatenkalkül* sent with [Letter L9]. He has not yet talked with the *Herren* in Münster about it, since he will be there again only in about 10 days, when the lecture period begins. Behmann's ideas in [Beh31] have always interested him very much, but they did not have received greater attention because so far they have just been program and it did not come to a precise calculus based on them. Although in the recent manuscript the ideas are explicated further in some respects, the situation has not much changed. Ackermann continues to describe the conditions that must be satisfied to allow an exact discussion about the ideas: (1) An overview on all used symbols – which would be easy to satisfy; (2) An exact specification of the combinations of symbols that should be considered as meaningful; (3) The formal rules and base forms that should be used. He refers to his publication [Ack41], where he pointed out the difficulties that arise if one wants to pass from Behmann's ideas to a precise calculus and mentions that Behmann has never commented on them. It does not suffice to show that the paradoxes in the known way are avoided, but has to be ensured that they do not appear in modified form. Ackermann brings the related work by Church [Chu32; Chu33] again (see [Letter L7] ) to attention, which later showed to be contradictory by the paradox discovered by Kleene and Rosser. Ackermann proceeds with an example intended to be helpful for pointing out the difficulties that can arise if there is no precise version of all inference rules available. He concludes with noting that in the referenced work by Church there is already a restriction operator (*Limitator*), at least in connection with the universal quantifier, asking for clarification of the exact correspondence of Behmann's notation for this case with Church's and Ackermann's own in the mentioned work, which follows Church's.

- L12 Behmann to Ackermann, Bremen-Aumund, 6 December 1955, typescript, 6 pages. Carbon copies of two revised manuscript sections were originally enclosed.

Thanking Ackermann for his letter [Letter L11], Behmann writes that he has incorporated some of the addressed points essentially in form of explanatory footnotes. He then discusses the three particular points (1)–(3). About (2) he mentions that he succeeded to bring substitution algorithm (*Einsetzungsalgorithmus*) and at the same time and in immediate relationship the criterion for meaningfulness (*Sinnhaftigkeitskriterium*) into a much

<sup>49</sup> In 1955 Heinrich Scholz (1884–1956) was professor emeritus, Hans Hermes (1912–2003) and Gisbert Hasenjaeger (1919–2006) were professors, and Ackermann was honorary professor at *Westfälische Wilhelms-Universität Münster* with its renowned *Institut für mathematische Logik und Grundlagenforschung* founded in 1950 by Scholz and lead after his retirement 1953–66 by Hermes.

simpler and now completely determined form (*zwangsläufige Gestalt*). Accordingly, he changed the subject of his work to *Der Algorithmus der Einsetzung und der widerspruchsfreie Prädikatenkalkül*. Concerning (3) he is reluctant to fix a thorough axiomatization at that point, because he does not consider axiomatic representation as helpful for getting a first understanding of a subject. For him, a strong axiomatic pre-load (*Vorbelastung*) was always an obstacle to get deeper into the meaning and essence of a subject. He had studied [Ack41] at its time, but the difficult outer and personal circumstances as well as the wish to not just criticize but also give a positive image had prevented that he commented it. Ackermann's use of restriction differs from his, in particular because the former is through the addition of provability loaded with modality. Behmann conjectures a circularity, if the concept of provability is already presupposed at the introduction of the junction. He further expresses doubts that simple applications such as syllogisms based on empirical facts can be expressed with Ackermann's provability-based notion. Behmann explains the differences between Ackermann's and his standpoint: His own proceeding is crucially determined by starting from plain propositional calculus, with implication in the Stoic, or Fregean, resp., sense, without modal or proof technical pre-load. Only after its fixation quantification is added, then modality in form of modal operators, and finally meta logic with inclusion of proof theory. For a presentation of the principles of this building-up, he refers to [Beh55], still waiting for the proceedings to appear. He emphasizes that his notion of meaningfulness (*Sinnhaftigkeit*) is unrelated to decidability as provability or refutability (*Entscheidbarkeit (als Beweisbarkeit oder Widerlegbarkeit)*). In [Hal49, p. 80] he found an expression that corresponds to the translation of the restriction operator suggested by Ackermann for his and Church's notation. Further discussions concern the inclusion of the third truth value *meaningless (sinnlos)* into the calculus. Behmann mentions that he intends to include in his paper also a form of Russell's paradox that was earlier communicated by Ackermann. He can now present its resolution much shorter and clearer than in [Beh53]. Enclosing carbon copies of new Sections 6 and 7 of his paper, he asks Ackermann for judgment and transferal to his associates in Münster. He intends to send copies of the two sections as well as a copy of the present letter directly to Scholz.

## 32 Discussions Related to Second-Order Quantifier Elimination in Behmann's Further Correspondence

In this section, further discussions in Behmann's correspondence – beyond that with Ackermann – with immediate relationship to his work on the decision and elimination problem are summarized. They are presented chronologically, headed by the respective correspondence partners, Bertrand Russell, Rudolf Carnap, Heinrich Scholz and Alonzo Church.

### 32.1 Bertrand Russell

Behmann writes on 8 August 1922 in English to Bertrand Russell [BehNL, Kasten 2, I 60], sending him an offprint of [Beh22a]. He sketches the role of Russell’s work with respect to his dissertation, whose theme was proposed by Hilbert. Leaving this subject, Behmann continues: “A I already remarked, my article in the *Mathematische Annalen* follows another way. Notwithstanding my statement of its purpose and character in § 1 I beg leave for a few additional words. It was what I call the Problem of Decision, formulated in the said paragraph, that induced me to study the logical work of Schröder. And I soon recognized that in order to solve my particular problem, it was necessary first to settle the main Problem of Schröder’s Calculus of Regions, his so-called Problem of Elimination. And – here I quote a sentence from a lecture of mine held before the Göttinger Mathematische Gesellschaft – “I believe it to be a very lucky circumstance that now an opportunity presents itself to embrace the earlier investigation relevant to that topic, questionable, especially as regards the form of presentation, and of difficult access as they are, under a new, uniform, and valuable point of view, thus saving from oblivion a great deal of profound and hard work of thought.” – Indeed, the chief merit of the said problem is, I daresay, due to the fact that it is a problem of fundamental importance on its own account, and, unlike the application of earlier Algebra of Logic, not at all imagined for the purpose of symbolic treatment, whereas, on the other hand, the only means of any account from its solution are exactly those of Symbolic Logic. – But I wish to avoid anticipation.” (Parts of this letter are also quoted in [Man99]).

On 16 September 1922 Bertrand Russell [BehNL, Kasten 2, I 60] replies that, so far, he not have had time to read Behmann’s paper carefully, but notes “I see that you have a symbolism very admirably adapted to your subject”. He brings Sheffer’s article [She13] to attention and explains how “the logic of propositions can be developed with only *incompatibility* (not-p or not-q), instead of both negation + disjunction.” (that is, Sheffer’s stroke *NAND*) or equally well “(NOT-p and NOT-q)” (*NOR*).

### 32.2 Rudolf Carnap

In his postcard dated 20 November 1922 [BehNL, Kasten 1, I 10] Carnap thanks Behmann for sending him an offprint of [Beh22a]. He writes that it did interest him very much and asks Behmann for sending a copy to Gerhards<sup>50</sup>.

In a letter dated 19 February 1924 [BehNL, Kasten 1, I 10], Carnap relates Behmann’s work on the decision problem for relations to Behmann’s refusal to contribute to the development of a symbolism that is based on Russell’s. Carnap writes (in translation): “If you really succeed to solve the decision problem so far that it also includes the of theory of relations (*Beziehungslehre*), and in particular also that which uses constant non-logical relations, this would be a very lucky and valuable progress and extraordinarily useful for my works.

<sup>50</sup> Karl Gerhards (1888–1957), German philosopher, mathematician and physicist.

However, I believe that the development until practical applicability will still need a lot of time. Thus, *for the time being* I still want to stick to Russell's symbolism." Carnap can understand that Behmann, being now busy with the extension of his own symbolism, is not inclined to contribute to the appearance of the other symbolism.

### 32.3 Heinrich Scholz

As discussed in Sect. 3 and 4, Behmann explains in his letter from 27 December 1927 [BehNL, Kasten 3, I 63] to Heinrich Scholz his contribution to the decision and elimination problem. Here is an excerpt of the respective parts of the original German letter:

*Bezüglich des Entscheidungsproblems ist zu unterscheiden zwischen der Entscheidung innerhalb der rein aussagenlogischen – die nur Begriffe der linken Seite meiner Tabelle enthalten – und im Gesamtbereich der Begriffslogik. Wo die beiden Lösungen des elementaren Problems, die Verfahren der Einsetzung und der konjunktiven Normalform, zuerst erwähnt und systematisch dargestellt worden sind, ist mir nicht bekannt. Vielleicht kommt hier Whiteheads "Universal Algebra" in Frage. Nach meiner Meinung führt die elementare Aussagenlogik so zwangsläufig auf dieses Problem, dass seine Aufweisung und Erledigung wohl mit der ersten strengen Darstellung jener überhaupt zeitlich zusammenfallen wird. So weit ich mich erinnere, habe ich das Verfahren der Normalform – das andere steht ja in den PM – durch Hilbert, der sich ja schon vor dem Kriege selbständig mit dem Problem der symbolischen Darstellung der Logik, und zwar insbesondere der Aussagenlogik, beschäftigte, kennen gelernt und weiss gerade aus diesem Grunde hinsichtlich der Literatur dieses Punktes keine sichere Auskunft zu geben.*

*Was nun das allgemeinere Entscheidungsproblem betrifft – nicht zu verwechseln mit dem z.B. von Hessenberg in einer seiner Schriften in den Abh. d. Friesschen Sch. besprochenen Entscheidbarkeitsproblem (im Zusammenhang mit dem Paradoxon von Richard) –, so ist dieses ausdrücklich als solches meinen Wissens vor mir von niemandem behandelt worden. Wohl aber ist unabhängig von dieser Fragestellung oder doch zum mindesten ohne ihre ausdrückliche Erwähnung eine gewisse Teilaufgabe, das sogenannte Eliminationsproblem, das übrigens zugleich einem passend eingeschränkten Entscheidungsproblem äquivalent ist, behandelt worden, und zwar zunächst von den Amerikanern, insbesondere Peirce, und hierauf mit besonderer Liebe und Ausdauer von Schröder, und schliesslich hat es einen Spezialisten in Löwenheim gefunden, der in den Math. Annalen darüber mehrere Abhandlungen schrieb. Die wichtigste von ihnen ist "Ueber Möglichkeiten im Relativkalkül"; sie erschien, glaube ich, einige Jahre vor dem Kriege. L. ist dort schon zu wesentlichen Teilergebnissen meiner Abhandlung über das Entscheidungsproblem gelangt, allerdings – in einer weder mathematisch strengen noch hinreichend verständlichen Darstellung, so dass ich selber diese erst nach dem Erscheinen meiner eigenen Schrift richtig herausgelesen habe. Dazu gebraucht er eine ziemlich sonderbare Terminologie; was ich "Aussage des Bereiches A" nenne, heisst bei ihm "Zählgleichung", usw. Wie ich es mir erkläre, dass*

*man zu dieser Fragestellung gelangen und dabei an dem Entscheidungsproblem so völlig vorbeigehen konnte, habe ich in meiner Schrift über das Entscheidungsproblem S. 218 – 19 auseinandergesetzt.*

### 32.4 Alonzo Church

In his letter dated 15 April 1937 to Alonzo Church [BehNL, Kasten 1, I 11], Behmann comments seemingly an offprint sent earlier by Church, where a system by Quine is discussed. Behmann states that the idea as such to combine (*zusammenfassen*) several argument to complexes that can then be handled like uniform (*einheitliche*) arguments is rather obvious. He had applied it himself in [Beh27a] (see Sect. 25) to reduce the general elimination problem – as preliminary step (*Vorsutufe*) to the decision problem – for the case of universal individual quantifiers onto the sequence (22), p. 59. However, without making the principles of the reduction explicit. He considers the idea to use this possibility already for building-up propositional and predicate logic as new and valuable. But, he continues, it needs to be done in way such that natural associativity is preserved; Quine’s scruples on this should be countered by a reasonable formulation of the logical rules. (Behmann also remarks that the statement in [Beh27a] that elimination can be performed in general for that case does not hold.)

Church answers on 20 May 1937 [BehNL, Kasten 1, I 11]: “I believe that I am in substantial agreement with what you say about Quine. There is only one remark which it occurs to me to make. He does not introduce a law of associativity such as you suggest. With him the ordered pair  $(x, y)$  is introduced as an undefined concept;  $(x, (y, z))$  is not the same as  $((x, y), z)$ ; the ordered triad  $(x, y, z)$  is defined to be  $((x, y), z)$ , the ordered tetrad  $(x, y, z, t)$  to be  $((x, y), z, t)$ , and so on. This seems to me satisfactory; from certain points of view one could even go as far as to say that an associative law would be undesirable.”

In a much later letter from Behmann to Church, dated 30 January 1959, where the main topics are relationship of [Beh59] to lambda conversion, as well as, possibilities to simplify propositional formulas, Behmann sketches his method in [Beh61]: As first step there, by means of simple transformation rules – of the kind of Gentzen’s *Mischungsregel*, which, Behmann writes, he as already stated in 1922 (i.e. in [Beh22a]) – the totality of prime implicants is obtained in a deterministic way (*zwangsläufig gewonnen*). Also the name *innex* form is suggested in this letter for the result form of his method in [Beh22a], where quantifiers are propagated inwards (see p. 11, footnote 7).

## 33 Documents that could not be Located

The following documents would be relevant for the consideration of second-order elimination and are referenced in the Behmann’s bequest, but, so far could not be located:

1. The technical part of Behmann's transcript of his talk *Entscheidungsproblem und Logik der Beziehungen* on 23 September 1926 in Düsseldorf, which he sent in 1928 to Ackermann. See notes on [Manuscript M10] and [Letter L4].
2. A contribution by Behmann on the solution and elimination problem (*zum Problemkreis Auflösungs- und Eliminationsproblem*) in a memorial publication (*Festschrift*) for Ernst Schröder, mentioned in 1942 in the correspondence with Scholz.

Two issues where, so far, no further relevant documents could be found in the bequest are normalization with respect to given predicates, suggested in [Beh22a, p. 201] (see Sect. 20) and the actual influence of Löwenheim's work [Löw15] and possibly Skolem's papers [Sko19; Sko20] (with exception of Behmann's 1927 letter to Scholz see Sect. 3, 4 and 32.3).





Part VI

Conclusion



### 34 Concluding Remarks

The early results on the decision problem by Löwenheim, Skolem and Behmann 1915–22 included the identification of relational monadic formulas as decidable fragment of first-order logic, the development of a decision procedure for this fragment, as well as the first explicit statement of the decision problem itself by Behmann 1921. As became evident from our detailed inspection of Behmann’s Habilitation thesis from 1922, where he made precise and extended earlier work by Schröder, these early investigations of the decision problem were closely tied to the problem of second-order quantifier elimination. The basic connection there is that a relational monadic formula can be decided by eliminating all its predicates one by one.

The decision problem has been much researched since then, with several monographs [Ack54; Sur59; DG79; BGG97], where in particular the last cited one gives a comprehensive survey, also on the history and literature. Despite many “theoretical” results, the development of calculi that decide specific classes is still an active issue of research in automated deduction.

The elimination problem was brought to larger attention in computational logic in the 1990s with the development of two algorithms, the resolution-based SCAN [GO92] and DLS [Sza93]. The latter initiated the so-called *Ackermann approach*, which is, like Behmann’s method, based on equivalence preserving formula rewriting. While DLS explicitly involves an essential idea from Ackermann’s 1935 paper [Ack35a], SCAN is in part a re-discovery of another technique from that paper [NOS99]. Early considered applications of second-order quantifier elimination in computational logic were the computation of first-order correspondence properties of modal formulas [GO92; CGV06; Sch12] and non-monotonic reasoning by computing circumscription [DLS97], which can be extended to model various semantics for logic programming [Wer10] and to the computation of abductive explanations [KKT98] with respect to these semantics [Wer13]. Since the mid 2000s, variants of second-order quantifier elimination like uniform interpolation, forgetting and projection receive great interest as operations for the processing of description logic knowledge bases, e.g. [GLW06; LW11; LK14]. A number of specialized elimination methods for particular description logics have been developed, where advanced ones explicitly relate to SCAN and the Ackermann approach [KS13]. Another current activity related to second-order quantifier elimination is in SAT solving the investigation of formula simplifications that perform elimination of Boolean variables in restricted ways [Bie04].

In contrast to the decision problem, there is only a single monograph [GSS08] on the elimination problem, with focus on modern developments. Some early and fundamental results such as the success of elimination on relational monadic formulas and Behmann’s rewriting-based method as a blueprint of the more advanced DLS apparently came to surface again only recently [Wer15]. As shown there, one can even today learn from Behmann’s method possible improvements of modern methods such as DLS, the success on relational monadic formulas emerges as a useful completeness property of elimination methods, and on a

closer look the seemingly not very expressive class of relational monadic formulas (*Klassenlogik*, as it was called at Behmann’s time), when considered together with second-order quantification, shows interesting relationships to description logics, the logics regarded today as adequate to represent concept (or *class*) ontologies.

Behmann’s primary concern was the decision problem. In his correspondence with Ackermann it can be observed that he talks about the decision problem, even if the discussed method actually performs elimination, whereas Ackermann often speaks about the elimination problem. The decision problem was first stated explicitly by Behmann, the elimination problem, in contrast, was also investigated earlier by others, in particular Schröder and Löwenheim. Nevertheless, the methods developed by Behmann decide formulas by performing elimination. He was dissatisfied with later decidability results by Bernays, Ackermann, Schönfinkel and Schütte that reside on satisfiability for domains with finite cardinalities determined from syntactic structure, since these results only lead to primitive methods, not suited for practical application. Thus, Behmann’s main concern was actually not just the decision problem: it was the problem of finding *practically applicable decision methods*.

This places his work into the context of computer science. He used a relatively small set of essentially syntactic tools, which let his methods smoothly fit into modern computational logic: Rewriting formulas such that equivalence is preserved or to entailed formulas. Moving arguments inward and outward of terms. Adding auxiliary definitions. Distribution among connectives. Propagating quantifiers inward and outward. Various ways of normalization, including generalizations of disjunctive and conjunctive normal form and Boolean combinations of specific basic forms. Skolemization and un-Skolemization. And, as outlined in a manuscript and the correspondence with Ackermann, representing infinite sets of formulas by schematic formulas with superimposed graph structure. In view of his later works, quantifier restriction and variants of lambda conversion have to be added.

Some of the material in this report should be useful for further investigations in the technical-historical realm, such as the clarification of the exact relationship between the works of Löwenheim [Löv15], Skolem [Sko19; Sko20] and Behmann [Beh22a], or an in-depth investigation of Ackermann’s resolution-based method for second-order quantifier elimination in [Ack35a], in presence of the related unpublished works by Behmann summarized here, related work by Craig [Cra60], modern resolution-based elimination methods [GO92; Gor+04] and fixpoint techniques as used to specify logic programming semantics [EK76] and in elimination [NS98].

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