Second-Order Characterizations of Definientia in Formula Classes

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1. Introduction

Definientia

- A **definition of G in terms of S within F** is a formula $(G \leftrightarrow X)$ s.t.
  1. $F \models (G \leftrightarrow X)$, and
  2. X contains only symbols from S
- G is the **definiendum**

  X is the **definiens**
- This applies also to first-order logic:

  If there are no free variables in F, then
  $F \models \forall \mathbf{x}(G(\mathbf{x}) \leftrightarrow X(\mathbf{x}))$ iff
  $F \models G(\mathbf{x}) \leftrightarrow X(\mathbf{x})$
- We are interested in **computing definientia** X for given F, G and S

An Application: Definientia as Exact View-Based Query Rewritings

[Halevy 01, Calvanese* 07, Marx 07, Nash* 10, Bárány* 13, W 14a]

- Given:

$\mathit{DBSymbols}$	$\{a\}$
$\mathit{ViewSymbols}$	$\{p, q\}$
$\mathit{ViewSpec}$ in terms of $\mathit{DBSymbols} \cup \mathit{ViewSymbols}$	$(p \leftrightarrow a) \land (q \leftrightarrow a)$
Query in terms of $\mathit{DBSymbols}$	a

- Compute a $\mathit{Rewriting}$ of Query in terms of $\mathit{ViewSymbols}$ s.t. for all DB:

$$\mathit{DB} \land \mathit{ViewSpec} \models \mathit{Rewriting} \quad \text{iff} \quad \mathit{DB} \models \mathit{Query}$$

- (Under certain assumptions on $\mathit{ViewSpec}$), the $\mathit{Rewritings}$ are the **definientia** of Query in terms of $\mathit{ViewSymbols}$ within $\mathit{ViewSpec}$

$$\begin{array}{lcl}
(p \leftrightarrow a) \land (q \leftrightarrow a) & \models & a \leftrightarrow (p \land q) \\
(p \leftrightarrow a) \land (q \leftrightarrow a) & \models & a \leftrightarrow p \\
(p \leftrightarrow a) \land (q \leftrightarrow a) & \models & a \leftrightarrow q \\
(p \leftrightarrow a) \land (q \leftrightarrow a) & \models & a \leftrightarrow (p \lor q)
\end{array}$$

$$\begin{array}{rcr}
a \land (p \leftrightarrow a) \land (q \leftrightarrow a) \models p & \text{iff} & a \models a \\
\neg a \land (p \leftrightarrow a) \land (q \leftrightarrow a) \models p & \text{iff} & \neg a \models a
\end{array}$$

Addressed Question

- Definientia **in terms of a given set of predicates** can be characterized **semantically by second-order formulas**
  - They can be **computed** by **second-order quantifier elimination**

    aka computation of **forgetting** and **uniform interpolants** [Doherty* 97, Gabbay and Ohlbach 92, Gabbay* 08, Ghilardi* 06, Konev* 09, Koopmann and Schmidt 13]
- It seems useful to compute definientia that are **in a given formula class** (like Horn or Krom formulas)
- **"Determinacy"** is investigated in database research [Segoufin and Vianu 05, Marx 07, Nash* 10, Bárány* 13]

  For $Query$, $ViewSpec$ in particular **formula classes**:
  - is the existence of an exact rewriting (definiens) decidable?
  - what formula class contains all exact rewritings (definientia)?

Can definientia in given formula classes be characterized by second-order formulas?

Two Basic Approaches

1. Characterizations **based on semantic properties**, such as the model intersection property for Horn formulas

2. Modeling syntactic characterizations by an auxiliary **"meta-level"** vocabulary

Can definientia in given formula classes be characterized by second-order formulas?

2. Toolkit: Classical Logic + Second-Order Operators

Classical Logic + Second-Order Operators

- We start with an **underlying classical logic**, e.g., first-order or propositional
- It is extended by **second-order operators**, e.g., predicate quantification or Boolean quantification

$$\exists q\,(p \rightarrow q) \land (q \rightarrow r)$$

For propositional logic: $\exists p\, F \;\equiv\; F[p \mapsto \textsc{true}] \lor F[p \mapsto \textsc{false}]$

- The associated computation is **second-order operator elimination**: computing an equivalent formula without second-order operators

$$\exists q\,(p \rightarrow q) \land (q \rightarrow r) \;\equiv\; p \rightarrow r.$$

Forgetting, Projection, Uniform Interpolation

	Second-order operator	Elimination aka
	$\exists q\, F$	Predicate quantifier elimination Boolean variable elimination
$\equiv$	$\mathrm{forget}_{\{q\}}(F)$	Computation of **forgetting**
$\equiv$	$\mathrm{project}_{\{p,r\}}(F)$	Computation of **projection** **Uniform interpolation**
$\equiv$	$\mathrm{forget}_{\textsc{AllPredicates}-\{p,r\}}(F)$	
$\equiv$	$\mathrm{project}_{\textsc{AllPredicates}-\{q\}}(F)$	

Considering Polarity: Literal Forgetting, Literal Projection

[Lang* 03, W 08]

- We generalize the first argument of forgetting and projection to a **set of ground literals**, called **scope**

  Effects on just **positive** or **negative** predicate occurrences can be expressed

  Literal forgetting and **literal projection** are now our basic operators

Let $F = (p \rightarrow q) \land (q \rightarrow r)$

$$\begin{aligned} & \mathrm{forget}_{\{\neg q\}}(F) \\ \equiv \quad & \mathrm{project}_{\{p,q,r,\neg p,\neg r\}}(F) \\ \equiv \quad & (p \rightarrow q) \land (p \rightarrow r) \end{aligned}$$

An interpretation is a set of ground literals, containing each ground atom either positively or negatively.

$I \models \mathrm{project}_S(F)$ iff$_{\text{def}}$ There exists a J s.t. $J \models F$ and $J \cap S \subseteq I$.

$\mathrm{forget}_S(F) \stackrel{\text{def}}{=} \mathrm{project}_{\text{ALLGROUNDLITERALS} \setminus S}(F)$.

Scope-Determined Circumscription

- Interpretations can be **partially ordered** according to the subset relationship between the set of ground atoms that they satisfy

$$\{p(a), \neg p(b), \neg q(a), \neg q(b)\} \;\leq\; \{p(a), p(b), q(a), \neg q(b)\}$$

- **Predicate circumscription** allows to characterize the set of models of a formula that are **minimal** w.r.t. this ordering and generalizations where
  - only extensions of specified predicates are compared
  - comparison requires that extensions of specified predicates are equal

  [McCarthy 80, Lifschitz 94, Doherty* 97]

- The **second-order operator** $\mathrm{circ}_S(F)$ can express these variations, generalized to model maximization [W 12]

$$\mathrm{circ}_{\{p,q\}}(p \vee q) \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$$

$I \models \mathrm{project}_S(F)$	iff$_{\mathrm{def}}$	There exists a J s.t. $J \models F$ and $J \cap S \subseteq I$.
$I \models \mathrm{raise}_S(F)$	iff$_{\mathrm{def}}$	There exists a J s.t. $J \models F$ and $J \cap S \subset I \cap S$.
$\mathrm{circ}_S(F)$	$\stackrel{\mathrm{def}}{=}$	$F \wedge \neg \mathrm{raise}_S(F)$.

Notation for Aboutness

- That F is **"about"** S, or **"in scope"** S is written

$$F \Subset S$$

Let $F = p \vee \neg q \vee (r \wedge \neg r)$

$$\begin{array}{lcl} F & \Subset & \{p, \neg q\} \\ F & \Subset & \{p, q, r, s, \neg p, \neg q, \neg r, \neg s\} \\ F & \not\Subset & \{p\} \end{array}$$

$F \Subset S$ iff$_{\text{def}}$ $F \equiv \text{project}_S(F)$.

Globally Strongest Necessary and Weakest Sufficient Condition

- The **globally strongest necessary condition** of G on S within F is

  $$\text{the strongest } X \Subset S \text{ s.th. } (F \land G) \models X$$

  It can be expressed by a second-order operator

  $$\mathrm{gsnc}_{\{p\}}((q \to p),\, q) \;\equiv\; p$$

- The **globally weakest sufficient condition** of G on S within F is

  $$\text{the weakest } X \Subset S \text{ s.th. } (F \land X) \models G$$

  It can be expressed by a second-order operator

  $$\mathrm{gwsc}_{\{p\}}((p \to q),\, q) \;\equiv\; p$$

- The analog concepts in [Lin 01] are not unique modulo equivalence. See also [Doherty* 01, W 12]

Let $\overline{S}$ denote the set of the complements of the members of scope S.

$$\mathrm{gsnc}_S(F, G) \quad \overset{\text{def}}{=} \quad \mathrm{project}_S(F \land G).$$
$$\mathrm{gwsc}_S(F, G) \quad \overset{\text{def}}{=} \quad \neg\mathrm{project}_{\overline{S}}(F \land \neg G).$$

Definientia, Definability in Terms of Second-Order Operators

Recall: A **definition** of G in terms of S within F is a formula $(G \leftrightarrow X)$ s.t. (1.) $X \Subset S$, and (2.) $F \models G \leftrightarrow X$. G is the **definiendum**, X is the **definiens**

- **Definientia** are exactly those formulas in the scope that are

  between the GSNC and the GWSC

Let $F = (p \leftrightarrow a) \land (q \leftrightarrow a)$, let $S = \{p, q, \neg p, \neg q\}$

$$\mathrm{gsnc}_S(F, a) \equiv (p \land q) \begin{array}{c} \models\ p\ \models \\ \models\ q\ \models \end{array} (p \lor q) \equiv \mathrm{gwsc}_S(F, a)$$

- **Definability** (existence of a definiens) holds iff

  the GSNC entails the GWSC

$\textsc{isDefiniens}(X, G, S, F)$ iff$_{\mathsf{def}}$ $X \Subset S$ and $\mathrm{gsnc}_S(F, G) \models X \models \mathrm{gwsc}_S(F, G)$.

$\textsc{isDefinable}(G, S, F)$ iff$_{\mathsf{def}}$ $\mathrm{gsnc}_S(F, G) \models \mathrm{gwsc}_S(F, G)$.

$\mathrm{gsnc}_S(F, G) \stackrel{\mathsf{def}}{=} \mathrm{project}_S(F \land G)$.

$\mathrm{gwsc}_S(F, G) \stackrel{\mathsf{def}}{=} \neg\mathrm{project}_{\overline{S}}(F \land \neg G)$.

So Far we Have:

- Second-order operators for

literal forgetting	$\mathrm{forget}_S(F)$
literal projection	$\mathrm{project}_S(F)$
predicate circumscription	$\mathrm{circ}_S(F)$
globally strongest necessary condition	$\mathrm{gsnc}_S(F, G)$
globally weakest sufficient condition	$\mathrm{gwsc}_S(F, G)$

- Characterizations in terms of second-order operations for

aboutness	$F \Subset S$
definiens	$\textsc{isDefiniens}(X, G, S, F)$
definability	$\textsc{isDefinable}(G, S, F)$

3. Horn Formulas and Horn Upper Bounds

Horn Formulas and the Least Horn Upper Bound

- The **least Horn upper bound** of a given formula is the **strongest Horn formula that is weaker than or equivalent** to the given formula [Selman and Kautz 91, Kautz* 95]
  - It is unique up to equivalence
  - It is equivalent to the **conjunction of all prime implicates that are Horn** [Selman and Kautz 91]
  - Let $\mathrm{lhub}(F)$ denote the **least Horn upper bound** of F
  - $\mathrm{lhub}(F)$ can characterized **semantically** as the strongest formula that is weaker than or equivalent to F and closed under "model intersection" [McKinsey 43, Dechter and Pearl 92]

    Closure under "model intersection" can be characterized by predicate quantification, but $\mathrm{lhub}(F)$ seems to require further means

"Filled" Horn Upper Bound

[W 14b]

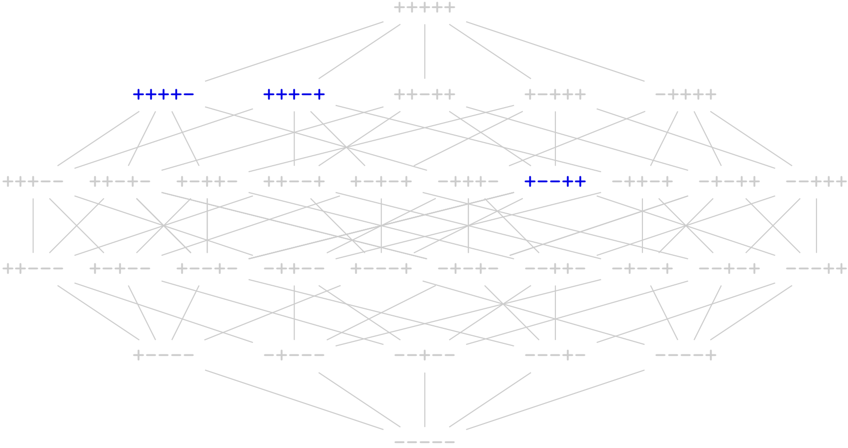
- Let $\mathrm{fhub}(F)$ denote the **"filled" Horn upper bound** of F, **another (possibly weaker) unique Horn upper bound**

$$
\begin{aligned}
\text{Let} \qquad F &= \mathrm{p} \land (\mathrm{q} \rightarrow \mathrm{r}) \land (\mathrm{s} \lor \mathrm{t}) \land \neg \mathrm{u}\\
\mathrm{lhub}(F) &\equiv \mathrm{p} \land (\mathrm{q} \rightarrow \mathrm{r}) \land \neg \mathrm{u}\\
\mathrm{fhub}(F) &\equiv \mathrm{p} \land \neg \mathrm{u}
\end{aligned}
$$

  - It can be **characterized just in terms of predicate quantification**, involving a second-order operator $\mathrm{diff}_S(F)$
  - The set of models of $\mathrm{fhub}(F)$, so-to-speak, completely **"fills"** the space "between" the greatest lower bound and the models of F

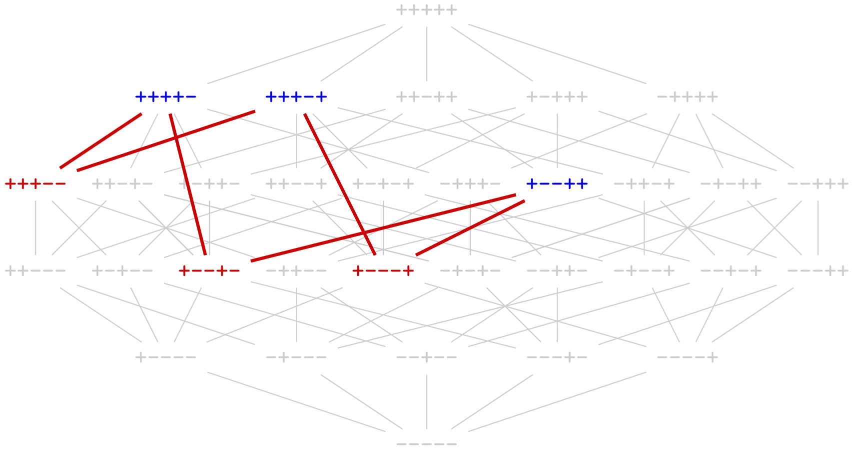
$I \models \mathrm{project}_S(F)$	iff_{def}	There exists a J s.t. $J \models F$ and $J \cap S \subseteq I$.
$I \models \mathrm{diff}_S(F)$	iff_{def}	There exists a J s.t. $J \models F$ and $J \cap S \not\subseteq I$.
$\mathrm{glb}(F)$	$\stackrel{\text{def}}{=}$	$\mathrm{circ}_{\mathrm{NEG}}(\neg \mathrm{diff}_{\mathrm{NEG}}(F))$.
$\mathrm{fhub}(F)$	$\stackrel{\text{def}}{=}$	$\mathrm{project}_{\mathrm{POS}}(\mathrm{glb}(F)) \land \mathrm{project}_{\mathrm{NEG}}(F)$.

Illustration: Horn Upper Bounds



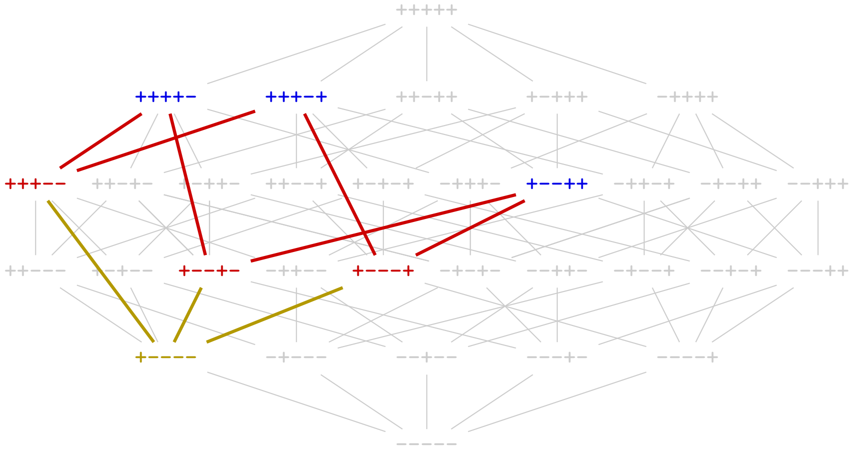
- Formula F

Illustration: Horn Upper Bounds



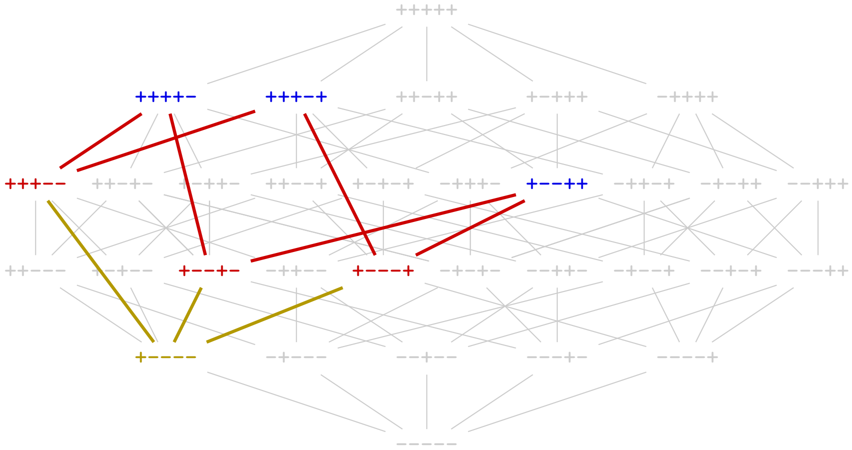
●	Formula	F
● ∪ ●	Model intersection step	$\mathrm{im}(F)$

Illustration: Horn Upper Bounds



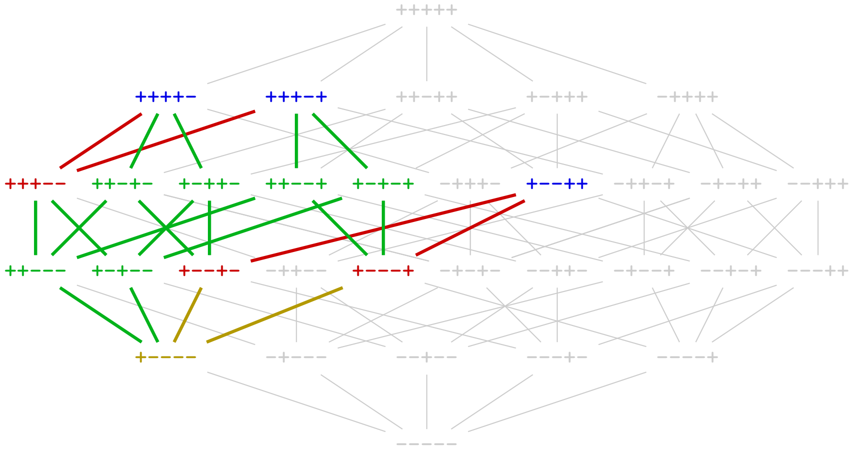
●	Formula	F
● ∪ ●	Model intersection step	$\mathrm{im}(F)$
● ∪ ● ∪ ●	Least Horn upper bound	$\mathrm{im}(\mathrm{im}(F)) = \mathrm{lhub}(F)$

Illustration: Horn Upper Bounds



●	Formula	F
● ∪ ●	Model intersection step	$\mathrm{im}(F)$
● ∪ ● ∪ ●	Least Horn upper bound	$\mathrm{im}(\mathrm{im}(F)) = \mathrm{lhub}(F)$
●	Greatest lower bound	$\mathrm{glb}(F)$

Illustration: Horn Upper Bounds



●	Formula	F
● ∪ ●	Model intersection step	$\mathrm{im}(F)$
● ∪ ● ∪ ●	Least Horn upper bound	$\mathrm{im}(\mathrm{im}(F)) = \mathrm{lhub}(F)$
●	Greatest lower bound	$\mathrm{glb}(F)$
● ∪ ● ∪ ● ∪ ●	Filled Horn upper bound	$\mathrm{fhub}(F)$

4. Expressing Definientia in Formula Classes

Considered Formula Classes and Shown Properties

- We consider the following formula classes:

SHORN	Formulas equivalent to a **Horn** formula
SCONATM	Formulas equivalent to a **conjunction of atoms**
SKROM	Formulas equivalent to a **Krom** formula

- A definiens in such a class $\mathcal{C}$ is called **$\mathcal{C}$-definiens**

- For each of these formula classes $\mathcal{C}$ we show
  - a **characterization of $\mathcal{C}$-definability**, that is, existence of a $\mathcal{C}$-definiens
  - a **representative $\mathcal{C}$-definiens**, that is, a (second-order) formula which is a $\mathcal{C}$-definiens under the sole precondition of $\mathcal{C}$-definability

[W 14b]

Definability and Definientia: SHORN

- Definability: *The least Horn upper bound of the GSNC entails the GWSC*
- Representative definiens: *The least Horn upper bound of the GSNC*

Theorem: G is $\mathcal{C}$-definable in terms of S within F iff

For $\mathcal{C} = \text{SHORN}$: $\text{lhub}(\text{gsnc}_S(F, G)) \models \text{gwsc}_S(F, G)$.

For $\mathcal{C} = \text{SCONATM}$: $\text{glb}(\text{gsnc}_{S \cap \text{POS}}(F, G)) \models \text{gwsc}_{S \cap \text{POS}}(F, G)$.

For $\mathcal{C} = \text{SKROM}$: G is SCONATM-definable in terms of $\text{KS}(S)$ within $(F \land \text{KD}(S))$.

Theorem: If G is $\mathcal{C}$-definable in terms of S within F, then the following formula is a $\mathcal{C}$-definiens:

For $\mathcal{C} = \text{SHORN}$: $\text{lhub}(\text{gsnc}_S(F, G))$.

For $\mathcal{C} = \text{SCONATM}$: $\text{fhub}(\text{gsnc}_{S \cap \text{POS}}(F, G))$.

For $\mathcal{C} = \text{SKROM}$: $\text{project}_S(\text{fhub}(\text{gsnc}_{\text{KS}(S)}(F \land \text{KD}(S), G)) \land \text{KD}(S))$

Definability and Definientia: SConAtm

- Here we consider the GSNC and the GWSC on the set of the positive literals in the specified scope
- Definability: *The greatest lower bound of the GSNC entails the GWSC*
- Representative definiens: *The filled Horn upper bound of the GSNC*
- Expressed just **by the introduced second-order operators**, which in turn are reducible to predicate quantification

Theorem: G is $\mathcal{C}$-definable in terms of S within F iff

For $\mathcal{C} = \text{SHORN}$: $\text{lhub}(\text{gsnc}_S(F, G)) \models \text{gwsc}_S(F, G)$.

For $\mathcal{C} = \text{SCONATM}$: $\text{glb}(\text{gsnc}_{S \cap \text{POS}}(F, G)) \models \text{gwsc}_{S \cap \text{POS}}(F, G)$.

For $\mathcal{C} = \text{SKROM}$: G is SConAtm-definable in terms of $\text{KS}(S)$ within $(F \land \text{KD}(S))$.

Theorem: If G is $\mathcal{C}$-definable in terms of S within F, then the following formula is a $\mathcal{C}$-definiens:

For $\mathcal{C} = \text{SHORN}$: $\text{lhub}(\text{gsnc}_S(F, G))$.

For $\mathcal{C} = \text{SCONATM}$: $\text{fhub}(\text{gsnc}_{S \cap \text{POS}}(F, G))$.

For $\mathcal{C} = \text{SKROM}$: $\text{project}_S(\text{fhub}(\text{gsnc}_{\text{KS}(S)}(F \land \text{KD}(S), G)) \land \text{KD}(S))$

Modeling Syntactic Characterizations by a Meta-Level Vocabulary

- Idea:
  1. define **"meta-level" symbols** for expressions
  2. restrict the "meta-level" symbols allowed in definientia

  Problem: Arbitrary combinations of **disjunctions** and **negations** of formulas would meet such restrictions

- **Negation and disjunction can be excluded with $\textsc{SConAtm}$-definientia**

Theorem: G is $\mathcal{C}$-definable in terms of S within F iff
For $\mathcal{C} = \textsc{SHorn}$: $\mathrm{lhub}(\mathrm{gsnc}_S(F, G)) \models \mathrm{gwsc}_S(F, G)$.
For $\mathcal{C} = \textsc{SConAtm}$: $\mathrm{glb}(\mathrm{gsnc}_{S \cap \mathrm{POS}}(F, G)) \models \mathrm{gwsc}_{S \cap \mathrm{POS}}(F, G)$.
For $\mathcal{C} = \textsc{SKrom}$: G is $\textsc{SConAtm}$-definable in terms of $\mathrm{KS}(S)$ within $(F \land \mathrm{KD}(S))$.

Theorem: If G is $\mathcal{C}$-definable in terms of S within F, then the following formula is a $\mathcal{C}$-definiens:
For $\mathcal{C} = \textsc{SHorn}$: $\mathrm{lhub}(\mathrm{gsnc}_S(F, G))$.
For $\mathcal{C} = \textsc{SConAtm}$: $\mathrm{fhub}(\mathrm{gsnc}_{S \cap \mathrm{POS}}(F, G))$.
For $\mathcal{C} = \textsc{SKrom}$: $\mathrm{project}_S(\mathrm{fhub}(\mathrm{gsnc}_{\mathrm{KS}(S)}(F \land \mathrm{KD}(S), G)) \land \mathrm{KD}(S))$

Definability and Definientia: $\mathrm{SK}\textsc{rom}$ – Auxiliary Formula

- $\mathrm{KD}(S)$ is the conjunction of the definitions of the "meta-level" atoms
  - empty, representing the empty clause, and
  - $\mathrm{clause}(L, M)$, representing nonempty Krom clauses
- $\mathrm{KS}(S)$ are the positive literals with the "meta-level" atoms

Assume a fixed total order $\leq$ on literals. Define:

$$\mathrm{KD}(S) \stackrel{\text{def}}{=} (\mathrm{empty} \leftrightarrow \bot) \land \bigwedge_{L,M \in S,\ L \leq M,\ L \neq \overline{M}} (\mathrm{clause}(L, M) \leftrightarrow L \lor M).$$

$$\mathrm{KS}(S) \stackrel{\text{def}}{=} \{\mathrm{empty}\} \cup \{\mathrm{clause}(L, M) \mid L, M \in S,\ L \leq M,\ L \neq \overline{M}\}.$$

Theorem: G is $\mathcal{C}$-definable in terms of S within F iff

For $\mathcal{C} = \mathrm{SH}\textsc{orn}$: $\mathrm{lhub}(\mathrm{gsnc}_S(F, G)) \models \mathrm{gwsc}_S(F, G)$.
For $\mathcal{C} = \mathrm{SC}\textsc{on}\mathrm{A}\textsc{tm}$: $\mathrm{glb}(\mathrm{gsnc}_{S \cap \mathrm{POS}}(F, G)) \models \mathrm{gwsc}_{S \cap \mathrm{POS}}(F, G)$.
For $\mathcal{C} = \mathrm{SK}\textsc{rom}$: G is $\mathrm{SC}\textsc{on}\mathrm{A}\textsc{tm}$-definable in terms of $\mathrm{KS}(S)$ within $(F \land \mathrm{KD}(S))$.

Theorem: If G is $\mathcal{C}$-definable in terms of S within F, then the following formula is a $\mathcal{C}$-definiens:

For $\mathcal{C} = \mathrm{SH}\textsc{orn}$: $\mathrm{lhub}(\mathrm{gsnc}_S(F, G))$.
For $\mathcal{C} = \mathrm{SC}\textsc{on}\mathrm{A}\textsc{tm}$: $\mathrm{fhub}(\mathrm{gsnc}_{S \cap \mathrm{POS}}(F, G))$.
For $\mathcal{C} = \mathrm{SK}\textsc{rom}$: $\mathrm{project}_S(\mathrm{fhub}(\mathrm{gsnc}_{\mathrm{KS}(S)}(F \land \mathrm{KD}(S), G)) \land \mathrm{KD}(S))$

Definability and Definientia: $\text{SK}\textsc{rom}$

- Definability: *Definable by a conjunction of the "meta-level" atoms*
- Representative definiens: *Take the representative definiens as conjunction of the "meta-level" atoms and convert it by projection to the original vocabulary*

Theorem: G is $\mathcal{C}$-definable in terms of S within F iff

For $\mathcal{C} = \text{SH}\textsc{orn}$: $\text{lhub}(\text{gsnc}_S(F, G)) \models \text{gwsc}_S(F, G)$.

For $\mathcal{C} = \text{SC}\textsc{on}\text{A}\textsc{tm}$: $\text{glb}(\text{gsnc}_{S \cap \text{POS}}(F, G)) \models \text{gwsc}_{S \cap \text{POS}}(F, G)$.

For $\mathcal{C} = \text{SK}\textsc{rom}$: G is $\text{SC}\textsc{on}\text{A}\textsc{tm}$-definable in terms of $\text{KS}(S)$ within $(F \wedge \text{KD}(S))$.

Theorem: If G is $\mathcal{C}$-definable in terms of S within F, then the following formula is a $\mathcal{C}$-definiens:

For $\mathcal{C} = \text{SH}\textsc{orn}$: $\text{lhub}(\text{gsnc}_S(F, G))$.

For $\mathcal{C} = \text{SC}\textsc{on}\text{A}\textsc{tm}$: $\text{fhub}(\text{gsnc}_{S \cap \text{POS}}(F, G))$.

For $\mathcal{C} = \text{SK}\textsc{rom}$: $\text{project}_S(\text{fhub}(\text{gsnc}_{\text{KS}(S)}(F \wedge \text{KD}(S), G)) \wedge \text{KD}(S))$

5. Conclusion

Open(ed) Issues

Q1: How to **express the least Horn upper bound** – fixpoint extension?

Q2: What **first-order formula classes** correspond to semantic properties like closure under model intersection?

Q3: What further **formula classes and properties** can be handled?

Q4: Are there useful **properties of the second-order expressions** characterizing definability and definientia, also with respect to arguments from specific classes?

Q5: Are there relationships to works on **non-uniform interpolation**?

Q6: Can the characterizations be applied with **approximations** like GSNC and GWSC instead of definitions?

Q7: What about **computation** of definientia? Layers involved:

- manipulation on the operator level
- eliminating the second-order operators
- conversion to the actual syntactic form

Q8: Can the approach be **practically** used?

- implemented with *ToyElim* [W 13], suitable for tiny experiments
- support for nested forgetting missing in current DL systems

Summary

- Steps towards a **formalized** and **mechanizable** bridge between
  - **expressibility in formula classes** and
  - **expressibility in restricted vocabularies**, formulated essentially by predicate quantification
- Demonstration with propositional logic as basis for **conjunctions of atoms**, **Krom** formulas, and to some degree for **Horn** formulas

Appendix

Example: SConAtm-Definientia

- Let $F = (\mathrm{q} \rightarrow \mathrm{r} \lor \mathrm{s}) \;\land\; (\mathrm{t} \rightarrow \mathrm{q}) \;\land\; ((\mathrm{r} \lor \mathrm{s}) \land \mathrm{u} \rightarrow \mathrm{p}) \;\land\; (\mathrm{p} \rightarrow \mathrm{t} \land \mathrm{u})$
- Consider finding definientia of p within F, in terms of positive occurrences of the other atoms $S = S \cap \mathrm{POS} = \{\mathrm{q}, \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{u}\}$
- Then: $$\begin{aligned} \mathrm{gsnc}_S(F, \mathrm{p}) &\equiv \mathrm{q} \land \mathrm{t} \land \mathrm{u} \land (\mathrm{r} \lor \mathrm{s}). \\ \mathrm{gwsc}_S(F, \mathrm{p}) &\equiv \mathrm{u} \land (\mathrm{q} \lor \mathrm{r} \lor \mathrm{s} \lor \mathrm{t}) \end{aligned}$$
- None of both is equivalent to a conjunction of atoms
- By the theorem, there must exist a SConAtm-definiens: $$\mathrm{glb}(\mathrm{gsnc}_S(F, \mathrm{p})) \equiv (\mathrm{q} \land \mathrm{t} \land \mathrm{u} \land \neg\mathrm{p} \land \neg\mathrm{r} \land \neg\mathrm{s}) \models \mathrm{gwsc}_S(F, \mathrm{p})$$
- By the theorem, $\mathrm{fhub}(\mathrm{gsnc}_S(F, \mathrm{p}))$ is a SConAtm-definiens: $$\mathrm{gsnc}_S(F, \mathrm{p}) \models \mathrm{fhub}(\mathrm{gsnc}_S(F, \mathrm{p})) \equiv (\mathrm{q} \land \mathrm{t} \land \mathrm{u}) \models \mathrm{gwsc}_S(F, \mathrm{p})$$

References

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